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Note

A note on k -price auctions with complete information

Yair Tauman

*Faculty of Management, Tel Aviv University, Tel Aviv, Israel 69978
Department of Economics, State University of New York, Stony Brook, NY 11794, USA*

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Abstract

Under complete information, for $k \geq 3$, a seller of a single-unit k -price auction obtains higher revenue than in a first or second price auction when the valuations and the bids are discrete and no bidder is using a (weakly) dominated strategy.

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1. Introduction

In this note it is shown that if bidders in third- and lower-price auctions are confined to strategies that are not (weakly) dominated, and if their valuations are common knowledge, then the revenue of the seller in each one of these auctions exceeds that obtained in first- and second-price auctions.

The literature on k -price auctions, for $k \geq 3$, focused on symmetric bidders with independent private values and ignored (to the best of my knowledge) the synthetic case of complete information which has at least pedagogical value. The assumption that bidders do not use weakly dominated strategies implies, for $k \geq 2$, that bidders bid at least their valuations. This is supported by Kagel

E-mail address: tauman@post.tau.ac.il.

and Levine's (1993) experimental work which finds that 80% of all bids in third-price auctions exceed private values. Analytical study of k -price auctions, for $k \geq 3$, is provided in Monderer and Tennenholtz (1998, 2000) and in Wolfstetter (2001). In Monderer and Tennenholtz (1998) it is argued that symmetric bidders with independent private values who are risk seekers pay a higher price in a k -price auction, for $k \geq 3$, than in either a first- or second-price auction. In this note we show that this is basically true for all bidders if their valuations are common knowledge.

2. Model

Consider a k -price sealed bid auction in which a single unit is offered for sale to $n \geq k$ bidders. This auction is defined as follows. The bidders submit simultaneously bids in sealed envelopes. The object is awarded to the highest bidder (ties are resolved at random) who pays the k th highest bid for the object.¹ All other bidders pay zero.

Assume without loss of generality that the valuations of the n bidders are $v_1 \geq v_2 \geq \dots \geq v_n$ and these values are assumed to be common knowledge to all bidders. In particular, every bidder knows not only his value but the values of all other bidders as well.

We characterize the pure strategy Nash equilibria of the k -price auction for $k \geq 3$, under the assumption that bidders do not use weakly dominated strategies. Notice that under this assumption bidders bid truthfully ($b_i = v_i$) in a second-price auction and thus the seller obtains v_2 . The first-price auction does not have a pure strategy equilibrium, unless bids are discrete. If the smallest unit bid is ε (and $v_i = m_i \varepsilon$ for some integer m_i , $1 = i = n$) and if $v_1 > v_2$, then in every equilibrium outcome (in pure strategies, which are not weakly dominated) $b_1 = v_2$, $b_2 = v_2 - \varepsilon$, and the seller obtains again v_2 .

3. Result

Definition. An equilibrium of the k -price auction is sensible if no bidder uses a weakly dominated strategy. It is efficient if the winner has the highest valuation.

Proposition 1. For $k \geq 3$, every sensible pure strategy equilibrium of the k -price auction is efficient and the seller obtains a payoff π such that $v_2 \leq \pi \leq v_1$. Conversely, every π , $v_2 \leq \pi \leq v_1$, can be supported by a sensible and efficient equilibrium of the k -price auction, for every $k \geq 3$.

¹ To avoid ambiguity in case there are ties, let b_1, \dots, b_n be the n bids and let $b_{(1)}, \dots, b_{(n)}$ be a decreasing order of the bids. Then $b_{(k)}$ is the k th highest bid.

Proof of the Proposition.

Step 1. Let $k \geq 2$. Then, in every sensible equilibrium of the k -price auction, $b_i \geq v_i, i = i, \dots, n$.

Proof. The strategy $b_i = v_i$ weakly dominates any strategy b'_i such that $b'_i < v_i$. \square

Step 2. A strategy b_i such that $b_i \geq v_i$ is not dominated by any other strategy.

Proof. The strategy $b_i, b_i \geq v_i$ is a strictly better reply than any $b'_i, b'_i < b_i$, in the case where $b_j = b_i$ for some $j \neq i$ and $b_m < v_i$ for all $m \notin \{i, j\}$. It is a strictly better reply than any $b'_i, b'_i > b_i$, in the case where $b_m = \frac{1}{2}(b_i + b'_i)$ for all $m \neq i$. \square

Step 3. The winning price (the k th highest bid) cannot exceed v_1 . Otherwise, the winner is better off dropping his bid.

Step 4. Let $k \geq 3$ and let π be the k th highest bid. Then $\pi \geq v_2$ in every sensible equilibrium.

Proof. Suppose that $\pi < v_2$. By Step 1, $b_1 \geq v_1$ and $b_2 \geq v_2$. Thus, both bids are greater than π and, hence, bidders 1 and 2 will outbid each other indefinitely without affecting the winning price, a contradiction. \square

By Steps 3 and 4, in every sensible equilibrium of the k -price auction, for $k \geq 3, v_2 \leq \pi \leq v_1$.

Step 5. Let α be such that $v_2 \leq \alpha \leq v_1$. Then for every $k \geq 3$ there exists a sensible and efficient equilibrium of the k -price auction in which the seller obtains α .

Proof. Let $b_1 > v_1, b_2 = v_1$, and $b_m = \alpha$ for all $m \geq 3$. By Step 2, none of the strategies are dominated and clearly these strategies constitute a Nash equilibrium. \square

Step 6. Every sensible equilibrium is efficient.

Proof. Let j be the winner of the k -price auction, for $k \geq 3$, and suppose that $j > 1$. If π (k -highest bid) is smaller than v_1 then bidder 1, who bids $b_1 \geq v_1$, is one of the k -highest bidders. Thus, if he raises his bid above b_j he will win the object and will obtain $v_1 - \pi > 0$.

Suppose next that $\pi = v_1$. Suppose that bidder j is the winner and $j > 1$. Then it must be that $v_j = v_1$ and hence j has the highest valuation, as claimed. Note that 1 is the only winner if $v_1 > v_2$. \square

Remarks

Remark 1. Equilibrium points that are not sensible may not be efficient. Let $k = 3$, $b_1 = v_3$, $b_2 > v_1$, $b_3 = v_1$, and $b_m = v_m$ for $m \geq 4$. Bidder 2 wins and pays v_3 . Bidder 1 is the third highest bidder.

Remark 2. The proof of Step 4 assumes that the two bidders, 1 and 2, have no budget constraints. Otherwise, in addition to the previous equilibrium points, the seller now may end up with a payoff below v_2 . Suppose that x_i is the budget of bidder i and $x_i > v_i$. Then any π , $v_3 \leq \pi \leq v_1$ can be the outcome of a sensible equilibrium. If $v_2 < \pi \leq v_1$ then the supporting equilibrium points are just those described in the proposition above. If $v_3 < \pi \leq v_2$ then the winner is either bidder 1 or bidder 2, whoever has the highest budget. If $\pi = v_2$ and the budget of bidder 2 exceeds that of bidder 1 then each one of them can be a winner. If $\pi = v_3$ and $x_3 > x_2 > x_1$, then each of the three bidders can be a winner.

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