

Informational Matching

Silvio Rendón*

Universidad Carlos III de Madrid
Universitat Pompeu Fabra

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Abstract.- This paper analyzes the problem of matching heterogeneous agents in a Bayesian learning model. One agent gives a noisy signal to another agent, who is responsible for learning. If production has a strong informational component, a phase of cross-matching occurs, so that agents of low knowledge catch up with agents of higher knowledge. It is shown that

(i) a greater informational component in production makes cross-matching more likely;

(ii) as a new technology is mastered, production becomes relatively more physical and less informational;

(iii) greater dispersions of the ability to learn and transfer information make self-matching more likely; and

(iv) higher inequality leads to self-matching and to more inequality, whereas lower inequality leads to cross-matching, which can make less productive agents not only catch up with but also overtake more productive ones.

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*Tel.:+ 34 91 624 5746; fax: + 34 91 624 9875.

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1 Introduction

This paper analyzes the problem of matching heterogeneous agents in a Bayesian learning model. The informational, ‘dial-setting’, or ‘input-target’ production function has proven appropriate to model processes in which learning-by-doing plays an important role. Proposed by Prescott (1973) and Wilson (1975), it has been the subject of recent research by Jovanovic & Nyarko (1996) and has been applied to explain firm dynamics by Mitchell (2000) and labor market dynamics by Nagypál (2000). Jovanovic & Nyarko (1995) have also used it to model the transfer of human capital from older to younger agents. However, no one has tried to analyze the implications of matching agents with an informational production function. The matching problem proposed by Becker (1973) has been typically analyzed with standard symmetric production functions.

Over the recent years, increased sorting into homogeneous groups such as firms, marriages, schools, or neighborhoods has been receiving a great deal of attention, both empirically and theoretically (e.g. by Kremer (1997) and Fernández & Rogerson (2000)). The puzzle in need of an explanation is why in the past people with different productivities used to work together (cross-matching), while now groups are formed with people of similar productivities (self-matching).

In this paper, production has a physical and an informational component. The physical component is modelled as a standard production function of two inputs with a positive cross-derivative. As was shown by Becker (1973), in this context, if each agent supplies one of the inputs, it is optimal to have positive assortative matching or self-matching.

The informational component captures agents’ learning of the best way to produce, modelled as a process of Bayesian updating. Agents are ranked according to their productivities; however, their roles in production are different. While one agent, the engineer, collects noisy observations on an unknown parameter that characterizes the best way to produce, the other agent, the worker, produces and improves thereby her

knowledge on that parameter. To match them means to combine the engineer's ability to collect accurate information with the worker's productivity as captured in her knowledge of the unknown parameter. It is shown that the informational component unambiguously induces negative assortative matching or cross-matching. This paper coincides thus with recent papers of Anderson & Smith (2002) and of Chade (2002) in that dynamic matching of heterogeneous agents with learning introduces an incentive for negative assortative matching. The setups are, however, very different: in theirs agents learn about their initially imperfectly observed partners, whereas in this paper one agent supplies the other with noisy signals and the other agent learns about the best way to produce.

With an informational production function it is optimal to have a phase of cross-matching, so that workers of low knowledge catch up with those of higher knowledge. When information is spread across agents, that is, when learning is attained, the production component becomes relatively more important and self-matching becomes optimal. The switch in sorting optimality corresponds to production becoming relatively more physical and less informational over time.

An implication of this analysis is that economies and organizations that are in early stages of development would like to mix their people to produce learning and activate the growth process. Economies and organizations that already know the production process and therefore have exhausted their possibilities of learning are more oriented towards physical than informational matching. The same applies to the introduction of a new technology in an industry or in a firm: the learning stage is associated with mixing agents of different productivities, but when the new technology is mastered, self-matching becomes optimal. In a way, a phase of cross-matching corresponds to a stage of accumulation of knowledge, whereas a phase of self-matching is more agreeable with a stage where knowledge is applied to production.

A second implication is that increased differences within each type of agent favor mating of the likes. In other words, both increased dispersion of engineers' abilities

to transfer accurate information and increased dispersion of workers' productivities have the effect of making self-matching more likely. This coincides with Kremer & Maskin (1996) finding that a wider dispersion of skills explains the predominance of self-matching. A further implication is that self-matching in one period implies self-matching in the next period, whereas cross-matching can make less productive workers not only catch up with, but also overtake more productive ones.

The remainder of the paper is organized as follows. The next section describes the model; Section 3 presents two examples: one with a purely informational production function and one with both informational and physical components; conclusions are presented in Section 4.

2 Model

This section discusses the effects of matching two heterogenous agents when production has an informational and a physical component. The purpose is to find out the determinants of self-matching and cross-matching in an information-theoretic framework.

2.1 Agents

There are two types of agents:

The worker. Her productivity is x , $0 < x < \infty$. She applies her knowledge in production and experiences learning by doing, i.e., her productivity grows over time.

The engineer. His productivity is e , $0 < e < \infty$. He supplies the worker with a noisy signal on the best way to produce. The higher the productivity of the engineer, the more precise this signal. His productivity does not change.

There is a basic asymmetry between these two agents: the worker produces and accumulates knowledge on the production process; the engineer gathers new evidence on production outcomes and transfers this information to the workers. One can also think about these agents as a student (the worker), who is ranked by her knowledge and becomes more productive over time by increasing that knowledge, and a teacher (the engineer), who is ranked by his ability to bring accurate observations to the student, thereby training her and increasing her productivity.

2.2 Production Function

The production function is

$$q = f(x, e) - \alpha (y - z)^2.$$

The first term corresponds to the physical component of production as a function of the agents' match. This function is a standard physical production function with $f_x > 0$, $f_e > 0$, $f_{xe} > 0$. The second term corresponds to the informational component, a quadratic loss function in which y is the engineer's imperfect signal of the unknown parameter θ , z is the action determined by the worker, and $\alpha > 0$ denotes the importance of the informational component in the production function. The engineer gives a noisy information on θ to the worker:

$$y = \theta + \varepsilon,$$

where ε is a white noise with distribution $N(0, e^{-1})$, that is, better engineers give more precise signals on θ to the workers. The engineer's quality is contained in his ability for observing a signal and in his ability for communicating it to the worker. The worker has a prior distribution of θ :

$$\theta \sim N(\hat{\theta}, x^{-1}),$$

that is, better workers have more precise prior knowledge on θ .¹ The worker chooses z to

$$\min_z E [(\theta + \varepsilon - z)^2].$$

The optimal solution coming from the first order condition is $z^* = E\theta = \hat{\theta}$. Hence, expected production is

$$Eq = f(x, e) - \alpha (Var(\theta) + Var(\varepsilon)) = f(x, e) - \alpha (x^{-1} + e^{-1}) \quad (1)$$

In this function workers' and engineers' qualities enter in the production function both through the physical and through the informational component. Qualities of both agents are unidimensional, that is, more productive agents have absolute advantage, they are more productive in both the physical and the informational aspect of production.

2.3 Learning

The worker is responsible for learning from realized production. She updates her beliefs on the unknown parameter θ according to Bayes's rule:

$$\begin{aligned} \hat{\theta}' &= \frac{x}{x+e} \hat{\theta} + \frac{e}{x+e} y, \\ x' &= x + e, \end{aligned} \quad (2)$$

where the superscript denotes next period's variables. Knowledge is accumulated and fully appropriated by the worker, i.e., when the worker changes partner, her knowledge goes with her without depreciation. Any distinction between match-specific and general knowledge is absent in this analysis. As said above, the engineer's quality does not change.

¹A planner would have no way to rank workers according to their knowledge of the mean of θ if he or she also ignores true θ . In consequence, workers' productivity is only contained in the precision of their knowledge of θ , not in the location of the mean of θ .

2.4 Matching

A planner matches two workers to two engineers for two periods.² Let the best engineer be indexed by 1, so $\Delta e = e_1 - e_2 > 0$ always. The initial difference in workers at period 1 can be positive or negative:³ $\Delta x = x_1 - x_2 \stackrel{\leq}{\geq} 0$. This model is solved backwards by finding the optimal solution at period two and then using it to find the optimal solution at period one.

2.4.1 Period 2

The value function in the second period is the sum of the expected production of the two teams:

$$V'(x'_1, x'_2) = \max_{j \in \{1, 2\}} (Eq'_{1j} + Eq'_{2, 3-j}).$$

Define the maximand as $U'_j = Eq_{1j} + Eq_{2, 3-j}$ and the gain of assigning the best engineer to worker 1 (or of choosing $j = 1$) as $\Delta U' = U'_1 - U'_2$. Then using (1), we obtain

$$\begin{aligned} \Delta U' &= f(x'_1, e_1) + f(x'_2, e_2) - f(x'_1, e_2) - f(x'_2, e_1), \\ &\simeq f_{xe} \Delta x' \Delta e. \end{aligned} \tag{3}$$

Different assignments of engineers to workers make a difference only in the physical component, not in the informational component. As shown by Becker, when $f_{xe} > 0$, self-matching occurs, that is, it is optimal to assign the best engineer to the best worker in the second period, but it may not be optimal in the first period.

²A private solution with transferable utilities would give the same solution as the planner's solution. If utilities are non-transferable and there are no frictions, self-matching always occurs.

³One can assume that the initially best worker is indexed by 1. However, because workers' productivity evolves, this ordering may be reversed in period 2.

Consequently, the optimal solution in period 2 is the following:

$$j^* = \begin{cases} 1, & \text{if } \Delta x' > 0; \\ 2, & \text{if } \Delta x' < 0, \\ \text{indifferent} & \text{if } \Delta x' = 0. \end{cases}$$

The sign of $\Delta x'$ depends on the solution at period 1, denoted by i^* . Suppose that at period 1 worker 1 is the best, $\Delta x > 0$, then from (2) we know that

$$\begin{aligned} \text{if } i^* &= 1, \text{ then } \Delta x' = \Delta x + \Delta e > 0 \text{ and } j^* = 1; \\ \text{if } i^* &= 2, \text{ then } \Delta x' = \Delta x - \Delta e \begin{cases} > 0 \text{ and } j^* = 1; \\ < 0 \text{ and } j^* = 2. \end{cases} \end{aligned}$$

When $\Delta x < 0$, the solution is similar, the only difference being that the index 2 represents the best worker and 1 the worst; hence we can express the general solution as

$$j^* = \begin{cases} 1, & \text{if } |\Delta x| > \Delta e \text{ and } \Delta x > 0; \\ 2, & \text{if } |\Delta x| > \Delta e \text{ and } \Delta x < 0; \\ i^*, & \text{if } |\Delta x| < \Delta e. \end{cases}$$

The initial difference in workers' productivities determines then the optimal solution in the second period.

2.4.2 Period 1

The value function in the first period is

$$V(x_1, x_2) = \max_{i \in \{1,2\}} \{Eq_{1i} + Eq_{2,3-i} + \beta V'(x_1 + e_i, x_2 + e_{3-i})\}.$$

where β is the subjective discount rate. The maximand in the first period is $U_i = Eq_{1i} + Eq_{2,3-i} + \beta V'(x_1 + e_i, x_2 + e_{3-i})$. Using (1) and (2) we determine the

gain of choosing $i = 1$ when $\Delta x > 0$:

$$\begin{aligned} \Delta U &\simeq \begin{cases} \left[f_{xe} + \beta f'_{xe} \frac{\Delta e}{|\Delta x|} - \beta \alpha A \right] \Delta x \Delta e, & \text{if } |\Delta x| > \Delta e, \\ \left[f_{xe} + \beta f'_{xe} - \beta \alpha A \right] \Delta x \Delta e, & \text{if } |\Delta x| < \Delta e, \end{cases} \\ &\simeq \left(f_{xe} + \beta f'_{xe} \min \left[\frac{\Delta e}{|\Delta x|}, 1 \right] - \beta \alpha A \right) \Delta x \Delta e, \end{aligned} \quad (4)$$

where $A = \frac{e_1 + x_1 + x_2 + e_2}{(x_1 + e_2)(x_2 + e_1)(x_1 + e_1)(x_2 + e_2)} > 0$. Because of their respective cross-derivatives, the gain in utility of choosing the best engineer is always positive in the physical component and is always negative in the informational component. The optimal solution depends on the relative importance of these two components:

$$i^* = \begin{cases} 1, & \text{if } \alpha^* = \frac{1}{\beta A} \left(f_{xe} + \beta f'_{xe} \min \left[\frac{\Delta e}{|\Delta x|}, 1 \right] \right) > \alpha \text{ and } \Delta x > 0; \\ 2, & \text{otherwise.} \end{cases}$$

Cross-matching happens when the parameter weighting the informational component in the production function is above a certain threshold. A low informational component, on the contrary, facilitates self-matching. Notice that this result does not follow only from the informational production function contained in the quadratic loss function used, but from *its interaction with Bayes's rule*. In the second period the informational component is present and does not play any role in the optimal assignment of engineers to workers. It is only in the first period, when the planner takes into account the evolution of workers' productivities, that rewards to learning make cross-matching desirable. It is optimal to mix workers in order to spread information and give less informed workers the opportunity to catch up with more informed ones.

3 Examples and Comparative Statics

Following this analysis, there are two extreme cases. On the one hand, if the production function is purely physical, that is, if $\alpha = 0$, there is always self-matching

as in other physical matching models. On the other hand, if the physical component is not sensitive to the match, i.e., if the production function is purely informational, say $f(x, e) = 1$ implying $f_{xe} = 0$, there is always cross-matching, as it follows from (4). In other words, purely informational matching means cross-matching.

To illustrate the intermediate case, when the production function consists of an informational and a physical component, suppose now that the physical component is given by $f(x, e) = xe$. This was the production function used by Becker (1973) for analyzing positive assortative matching and by Kremer (1993) for studying connections between labor markets and development. As this physical production function has $f_{xe} = 1$, when $\beta = 1$ the threshold for the parameter weighting the informational component is

$$\alpha^* = \frac{1}{A} \left(1 + \min \left[\frac{\Delta e}{\Delta x}, 1 \right] \right), \forall \Delta x \geq 0$$

This threshold depends on the initial dispersion and on the levels of engineers' and workers' productivities, as stated in the following proposition.

Proposition 1 *For all Δe and for $\Delta x \geq 0$ self-matching is more likely,*

- (i) *the higher the dispersion of engineers' productivities, that is, $\frac{\partial \alpha^*}{\partial \Delta e} > 0$;*
- (ii) *the higher the dispersion of workers' productivities, that is, $\frac{\partial \alpha^*}{\partial \Delta x} > 0$, if $\Delta e > \sqrt{0.5}(x_2 + e_2)$;*
- (iii) *the higher the total precision of workers and of engineers, that is,*

$$\left. \frac{\partial \alpha^*}{\partial x_2} \right|_{\Delta x = \text{const}} = \left. \frac{\partial \alpha^*}{\partial e_2} \right|_{\Delta e = \text{const}} > 0.$$

Proof. In Appendix.

A greater dispersion of engineers' productivities, Δe , unambiguously increases α^* and makes thereby self-matching more likely. On the other hand, if the dispersion of engineers' productivities is greater than a given threshold, an increased dispersion of workers' productivities makes self-matching more likely. For both cases, more

dispersion of productivities, or more inequality within each type of agent, implies more self-matching, as in Kremer & Maskin (1996). Increasing productivities of either type of agent tends also to produce self-matching.

If $\Delta e > \sqrt{0.5}(x_2 + e_2)$, let $\alpha_l = \alpha^*|_{\Delta x=0}$ and $\alpha_h = \alpha^*|_{\Delta x=\Delta e}$, then, as illustrated in Figure 1, depending on Δx and α^* , there are three possible cases for a solution:

- Case I. $\alpha_l > \alpha$. There is always self-matching: the physical component of the production function is predominant.
- Case II. $\alpha_h > \alpha > \alpha_l$. There is self-matching for a high dispersion of workers' productivities, but cross-matching when dispersion is low. Cross-matching implies overtaking, that is, the worst worker in period 1 becomes the best in period 2.
- Case III. $\alpha > \alpha_h$. There is self-matching for a high dispersion of workers' productivities and cross-matching for a low dispersion of workers' productivities. If the dispersion of workers' productivities is lower than the dispersion of engineers' productivities, the worst worker overtakes the best one in the first period.

Figure 2 graphs the evolution of the dispersion of workers' productivities for these three cases. This dispersion evolves according to the following rule:

$$|\Delta x'| = \begin{cases} |\Delta x + \Delta e|, & \text{if } \alpha^* > \alpha; \\ |\Delta x - \Delta e|, & \text{otherwise.} \end{cases}$$

For relatively more physically intensive production functions, that is, in Case I, self-matching always happens and the dispersion of workers' productivities increases. For relatively more informationally intensive production functions, that is in Case II and Case III, a high dispersion of workers' productivities produces self-matching, which increases this dispersion even more. In turn, a low dispersion of workers' productivities leads to cross-matching which reduces this dispersion, so that the initially worst worker can not only catch up with but overtake the best one.

4 Conclusion

This paper has shown that for a class of production functions where information is important, it is optimal to have a phase of cross-matching to increase aggregate knowledge in an economy. It is by investing in increasing and spreading information over all agents that knowledge is accumulated. In a second phase, as learning is attained, the physical aspect of production becomes more important and, consequently, self-matching is observed. One can establish four basic conclusions:

- (i) a greater informational component in production makes cross-matching more likely;
- (ii) as a new technology is mastered, production becomes relatively more physical and less informational;
- (iii) greater dispersions of engineers' and of workers' productivities makes self-matching more likely; and
- (iv) inequality, as captured by the dispersion of workers' productivities, leads to more inequality, whereas a lower dispersion of workers' productivities, by making cross-matching more likely, may lead the worst worker not just to catch-up but to overtake the best worker.

Appendix. Proofs of Proposition 1

We can rewrite $A = \frac{2X+\Delta x+\Delta e}{(X+\Delta x+\Delta e)(X+\Delta e)(X+\Delta x)X}$, where $X = x_2 + e_2$. Then (i) follows:

$$\frac{\partial \alpha^*}{\partial \Delta e} = \begin{cases} \frac{\partial(1/A)}{\partial \Delta e} \left(1 + \frac{\Delta e}{\Delta x}\right) + \frac{1}{A\Delta x}, & \text{if } \Delta x \geq \Delta e; \\ 2\frac{\partial(1/A)}{\partial \Delta e}, & \text{if } \Delta x < \Delta e; \end{cases} > 0,$$

because $\frac{\partial(1/A)}{\partial \Delta e} = \frac{1}{A} \frac{X(3(X+\Delta x)+4\Delta e)+(\Delta x+\Delta e)^2}{(2X+\Delta x+\Delta e)(X+\Delta x+\Delta e)(X+\Delta x)} > 0$. ■

Similarly, (ii) is established by

$$\frac{\partial \alpha^*}{\partial \Delta x} = \begin{cases} \frac{\partial(1/A)}{\partial \Delta x} \left(1 + \frac{\Delta e}{\Delta x}\right) - \frac{1}{A} \frac{\Delta e}{\Delta x^2}, & \text{if } \Delta x \geq \Delta e; \\ 2\frac{\partial(1/A)}{\partial \Delta x}, & \text{if } \Delta x < \Delta e. \end{cases}$$

If $\Delta x < \Delta e$, clearly $\frac{\partial \alpha^*}{\partial \Delta x} > 0$, because $\frac{\partial(1/A)}{\partial \Delta x} = \frac{1}{A} \frac{X(3(X+\Delta e)+4\Delta x)+(\Delta x+\Delta e)^2}{(2X+\Delta x+\Delta e)(X+\Delta x+\Delta e)(X+\Delta e)} > 0$.

If $\Delta x \geq \Delta e$,

$$\frac{\partial \alpha^*}{\partial \Delta x} = \frac{\Delta x^3 + 2\Delta x^2(\Delta e + 2X) + \Delta x(\Delta e^2 + 3X\Delta e + 3X^2)}{A(2X + \Delta x + \Delta e)(X + \Delta x + \Delta e)(X + \Delta x)\Delta x} - \frac{2X\Delta e(X + \Delta e) + X\frac{\Delta e}{\Delta x}(\Delta e + X)(\Delta e + 2X)}{A(2X + \Delta x + \Delta e)(X + \Delta x + \Delta e)(X + \Delta x)\Delta x}.$$

For the particular case $\Delta x = \Delta e$, $\frac{\partial \alpha^*}{\partial \Delta x} \Big|_{\Delta x=\Delta e} = \frac{2\Delta e^2 - X^2}{A(X+2\Delta e)(X+\Delta e)\Delta e} > 0$, if $\Delta e > \sqrt{0.5}X$. This same condition implies that $\frac{\partial \alpha^*}{\partial \Delta x} > 0$, for $\Delta x \geq \Delta e$. ■

Notice that $\frac{\partial \alpha^*}{\partial x_2} \Big|_{\Delta x=const} = \frac{\partial \alpha^*}{\partial e_2} \Big|_{\Delta e=const} = \frac{\partial \alpha^*}{\partial X}$. Then (iii) follows:

$$\frac{\partial \alpha^*}{\partial X} = \frac{\partial(1/A)}{\partial X} \left(1 + \min \left[\frac{\Delta e}{\Delta x}, 1 \right] \right) > 0,$$

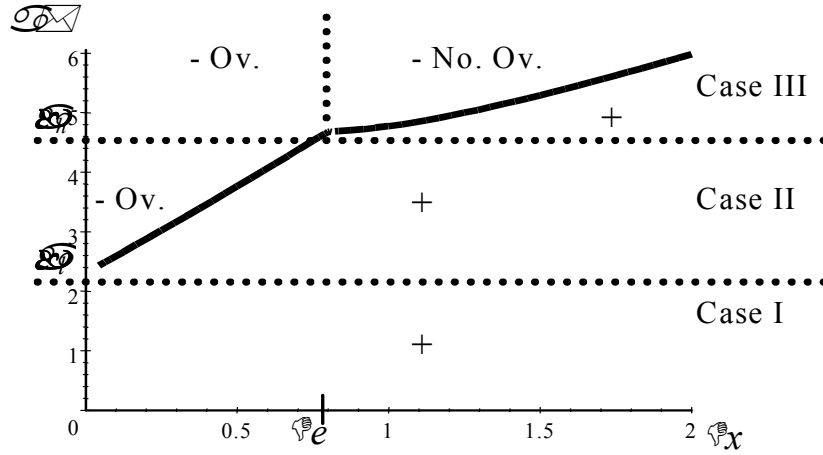
because $\frac{\partial(1/A)}{\partial X} = \frac{1}{A} \left[(X + \Delta x)^{-1} + (X + \Delta e)^{-1} + (X + \Delta x + \Delta e)^{-1} \right] + \frac{1}{A} \left[[\Delta x + \Delta e] [2X + \Delta x + \Delta e]^{-1} X^{-1} \right] > 0$. ■

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Figure 1: α^* as a function of Δx

$$(e_2 + x_2 = 1, \Delta e = 0.8).$$



- + : Self-Matching
- Ov. : Cross-Matching with Overtaking
- No Ov. : Cross-Matching without Overtaking

Figure 2: $\Delta x'$ as a function of Δx 