

BUDGET CONSTRAINTS AND DEMAND REDUCTION IN SIMULTANEOUS ASCENDING-BID AUCTIONS*

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The possibility, even if arbitrarily small, of binding budget constraints in simultaneous ascending bid auctions induces strategic demand reduction and generates significant inefficiencies. Under mild conditions on the distributions of the bidders' values, unconstrained bidders behave as if they were liquidity constrained, even as the probability that bidders are budget constrained goes to zero.

I. INTRODUCTION

MOST AUCTION THEORY IGNORES THE POSSIBILITY that bidders may be willing to pay more for goods on sale than the amount of money they have available, i.e., that bidders may be liquidity, or 'budget' constrained. Yet, liquidity constraints can play an important role in practice. For example, David Salant (Salant [1997] page 567), reporting on his experience in the bidding team at GTE during one of the FCC auctions for the sale of spectrum licenses, writes:

We were very concerned about how budget constraints could affect bidding. Most of the theoretical literature ignores budget constraints. In the MTA [Major Trading Area] auction, budget constraints appeared to limit bids.

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Salant also explains how, in order to formulate its strategy, the GTE bidding team used a simulation model in which possible budget levels of the different bidders entered as inputs.

In principle, if the bidders are interested in the objects for investment purposes (as in the case of spectrum licenses), and are able to finance their bids, liquidity constraints should not matter. However, frictions in capital markets often make the amount of available internal funds relevant. Moreover, even when external funding is available at profitable rates, a bidder may be reluctant to borrow from a third party, because this might require disclosure of private information. A bidder may even *choose* to be budget-constrained, in order to commit to a less aggressive bidding strategy and thus induce better outcomes in terms of final prices (see Benoît and Krishna [2001]). Finally, financial constraints can also emerge endogenously when bidders act as agents of financing principals (see e.g., Bolton and Scharfstein [1990]) and Holmström and Ricart i Costa [1986]). These considerations provide good theoretical and empirical reasons for analyzing the impact of budget constraints in auctions.

The introduction of budget constraints in theoretical models of auctions is fairly recent. Pioneering work in this area is due to Che and Gale, [1998] and [2000]. They have analyzed single-object environments in which buyers are privately informed about both their willingness and (possibly lower) ability to pay. Single-object second-price auctions with budget constrained bidders have also been studied by Fang and Parreiras, [2002] and [2003]. Zhèng [2001] studies a common value, (single object) first-price auction model in which the bidders can borrow at a given rate and default. Rhodes-Kropf and Viswanathan [2005] analyze single object first-price auctions with privately known values and budgets, in which the bidders can finance their bids with cash or securities. Multiple object auctions with budget constrained bidders have been studied in Benoît and Krishna [2001] under the assumption of complete information about both willingness to pay and budgets.

Our main goal in the present paper is to study the impact that the *possibility* of liquidity constraints has on the level of efficiency that can be achieved in simultaneous ascending bid auctions.¹ It is already known that, under mild conditions on the information and preference structure, these auctions provide bidders with ample opportunities for collusion.² Thus

¹ Since 1994, these types of auctions have been used repeatedly by the U.S. government to sell licenses for the use of parts of the electromagnetic spectrum. See Milgrom [2004].

² The basic idea behind collusive equilibria is that trying to win two objects yields less expected surplus than buying a single object at a relatively low price, see Brusco and Lopomo [2002]. Experimental results by Kwasnica and Sherstyuk [2001] corroborate our theoretical results. For a survey on recent experimental work on collusion in multiunit ascending bid auctions, see Sherstyuk [2002]. See also Cramton and Schwartz [2000], Klemperer [2002] or Milgrom [2004].

collusive behavior can already generate significant distortions from efficiency without the presence of liquidity constraints.

In the present paper we focus on ‘noncollusive’ equilibria (formally defined in Section 2), in which bidders compete as much as possible thus pushing the outcome as close as possible to the efficient one. We assume that each bidder’s willingness to pay for any object is not affected by the likelihood of winning any other object, so that, without budget constraints—i.e., when each bidder can bid up to her value on each object—the outcome of noncollusive bidding is always fully efficient.

We study a model with two identical objects and two bidders. Each bidder may or may not be liquidity constrained; that is, the amount of money that she can use to bid in the auction may be less than her willingness to pay for both objects. The possible presence of budget constraints has an obvious effect on efficiency: a constrained bidder is unable to win both objects when her value is above her opponent’s. Our analysis demonstrates that the possible presence of binding budget constraints also induces *unconstrained* bidders to strategically reduce their demand, in two ways: i) by behaving as if they were constrained, and ii) by reciprocating their opponents’ demand reduction. This happens because low-budget bidders must reduce their demand to one unit once the price of each item becomes half of their budget, hence the unconstrained bidders have the opportunity to either imitate this behavior, or reduce their demand before the prices arrive at their value.

Our main result is that the mere *possibility* (even if arbitrarily small) of binding budget constraints can reduce competition substantially and thus generate significant inefficiencies. More precisely, we show that for a large class of information structures, as long as the probability of binding budget constraints is positive, in any equilibrium there exists a set of positive measure of unconstrained types who behave as if they were constrained. Thus the objects may end up (inefficiently) divided between the two bidders even when both bidders are unconstrained. The probability of this event is amplified by a contagion effect, due to the fact that the incentive to imitate the behavior of constrained types increases with the probability that the opponent responds in kind.

The strategic demand reduction effect persists when the probability of having low-budget types goes to zero. Intuitively, this is because a high-budget bidder always reciprocates the opponent’s demand reduction with positive probability, and this in turn makes it profitable for some high budget types (with relatively low willingness to pay) to act as if they were budget constrained. We will show that this is always the case for a non-negligible set of types, and that, under mild conditions on the value distributions, the measure of this set remains bounded away from zero as the probability of having budget-constrained bidders goes to zero. In fact, for many distributions it turns out that *all* types, whether budget constrained or not, behave as if they were budget constrained.

This result is similar to the one found by Baliga and Sjöström [2004] for the ‘stag-hunt’ game: arbitrarily small probabilities that, for each player, the noncooperative strategy is dominant can eliminate the efficient equilibrium. The intuition for that result, in terms of a contagion effect triggered by arbitrarily small probabilities of types for which the ‘bad’ strategy is dominant, is similar to the one that we provide for our results.

The occurrence of strategic demand reduction has also been noted in multi-unit sealed-bid auctions with uniform pricing, in theoretical analyses by Ausubel and Cramton [2002] and Englebrecht-Wiggans and Kahn [1998]. The idea is also present in Wilson [1979]. Direct evidence of demand reduction in FCC spectrum auctions is also reported in Cramton [1998].³ In multi unit uniform-price auctions, the incentive to reduce demand is due to the concurrence of two features: i) the bidders have decreasing marginal willingness to pay, and ii) all items are sold at the same price. The incentive to reduce demand under these conditions comes from the ability to reduce the price paid for the infra-marginal units.

In environments with decreasing marginal willingness to pay, the use of a different auction format that allows for nonuniform pricing, such as the Vickrey auction or the ‘clinching’ mechanism proposed by Ausubel [2004], eliminates demand reduction and restores efficiency. In our model however the bidders have constant marginal willingness to pay. Therefore, unless the bidders coordinate on collusive equilibria, demand reduction is entirely attributable to the possibility of binding budget constraints. The direct effect, due to the inability to bid above one’s budget, already arises in the analysis of one-object auctions by Che and Gale [1998]. In our model, the presence of multiple units gives rise to the strategic effect as well, which amplifies the distortion from the first-best, even when the *ex-ante* probability of budget constraints is low. As we discuss in Section 5, this effect can be eliminated, thus limiting the distortions due to budget constraints to the direct effect, by making the buttons object-specific and banning reentry.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 discusses the effect of privately known budget constraints, and provides intuition for the demand reduction effect. In Section 4 we show that the demand reduction effect remains substantial even when the probability of having budget constrained bidders goes to zero. In Section 5 we discuss the effects of allowing for more than two budget levels and of various modifications of the auction rules. Section 6 concludes. The appendix collects all the proofs.

³ Direct evidence of demand reduction was seen in the nationwide narrowband auction. The largest bidder, PageNet reduced its demand from three large licenses to two, at a point when prices were still well below its marginal valuation for the third unit. PageNet felt that if it continued to demand a third license, it would drive up the prices on all the others to disadvantageously high levels.

II. MODEL AND PRELIMINARIES

There are two bidders and two identical objects. The objects are sold with a ‘continuous simultaneous ascending clock auction,’⁴ working as follows. Each bidder is given two buttons. The auction starts at a price of zero for both objects. The price rises at a constant speed as long as at least three buttons are pushed.

A bidder demands both objects at the current price by keeping both buttons pushed, and reduces her demand to $2 - k$, by releasing k buttons, $k = 1, 2$. Demand reduction is irrevocable: a button that has been released cannot be pushed again. This feature can be interpreted as a stylized version of the ‘activity rule’ used by the FCC in the spectrum auctions.⁵

The auction ends as soon as two or more buttons have been released. In order to avoid ‘open set problems,’ and make the game well defined, we use a version of the auction format proposed by Zhèng [2005], which allows for a ‘time to react’ when a bidder releases a button. More specifically, the auction rules can be described as follows. Let t denote the first time at which at least one bidder releases one or two buttons.

1. If, at time t , bidder i releases exactly one button and bidder $j \neq i$ does nothing, then the price stops increasing for an interval of time δ . During this period bidder j is given the opportunity to react, by releasing exactly *one* button. If she does so, the auction ends with each bidder buying one object for p_t . Otherwise the price resumes its upward movement, until at least one additional button is released. Let $t' > t$ denote the time when this happens. If only one additional button is released at t' , then each bidder buys a number of objects equal to the number of buttons that she is still pushing, otherwise each bidder buys one object.
2. If $k \geq 2$ buttons are released at time t , the auction ends at price p_t . When $k = 2$, each bidder buys a number of objects equal to the number of buttons that she is still pushing. When $k > 2$, each bidder wins exactly one object.

It is worth pointing out that the buttons are not object-specific: this formulation is equivalent to one where each bidder can resume bidding on any object even if she has not done so continuously since the beginning of the

⁴Clock auctions are defined in Milgrom [2004] as a variant of *simultaneous ascending auctions* introduced by the FCC in 1994. ‘The main difference between these [two auction formats] is that in the FCC design the bidders call the prices, whereas in a clock auction the auctioneer calls the prices (and posts them on a digital or analog clock). In Milgrom’s formulation both time and feasible bid sets are discrete. We have chosen the continuous time formulation because it simplifies the analysis.’

⁵The activity rules used by the FCC has been described as follows: ‘a bidder that places eligible bids for n units at round t cannot place bids for more than n units at any subsequent round $t' > t$.’ (Milgrom [2004]).

auction, as long as her bidding activity (the number of objects on which she is bidding) does not increase. Thus the present format, with the activity rule, is intermediate between object-specific irrevocable exit (once you stop bidding on a particular object you can never compete on that object again) and unrestricted reentry.

Each bidder $i = 1, 2$ is characterized by a two-dimensional type $\theta_i := (v_i, h_i)$, where v_i denotes her willingness to pay for one object and h_i denotes *half* of the maximum amount of money that she can spend in the auction. The utility of bidder i when she obtains n_i objects, and pays a total amount of m_i , is $n_i v_i - m_i$. We assume that a bidder can never bid above her budget, hence $m_i \leq 2h_i$. This can be justified either by assuming that the bids have to be backed by showing cash ('money on the table') or by assuming that there are stiff penalties for bankruptcy.

The four variables v_1, h_1, v_2, h_2 are distributed independently. For each $i = 1, 2$, h_i takes values $h_L < 1$ with probability λ , and $h_H > 1$ with probability $1 - \lambda$; and v_i has a density f which is strictly positive and differentiable on $[0, 1]$, and c.d.f. F . Thus bidder i 's type space is $\Theta_i := [0, 1] \times \{h_L, h_H\}$. We will often use the conditional c.d.f.

$$G(v_i) \equiv \frac{F(v_i) - F(h_L)}{1 - F(h_L)} 1_{[h_L, 1]}(v_i),$$

where $1_X(v_i)$ is the indicator function taking value 1 if $v_i \in X$ and zero otherwise.

To simplify the analysis, we also assume that budget-constrained bidders always have enough money to bid up to their value on a single object, i.e. $\frac{1}{2} < h_L$. This assumption can be relaxed without compromising our results, at the cost of increasing the number of sub-cases to be considered.

We focus on equilibria that rule out weakly dominated strategies. Our first lemma highlights the main behavioral implications of this restriction.

Lemma 1. The following continuation strategies are weakly dominated for bidder i :

1. reducing demand to zero when the price is $p_t < v_i$;
2. keeping a strictly positive demand when the price is $p_t > v_i$.

As mentioned in the introduction, the auction may have multiple equilibria. In Brusco and Lopomo [2002], we have shown that for any price $p \in [0, h_L)$, under a mild condition on the density f , there exists a 'collusive' equilibrium where each bidder bids on both objects up to the minimum between her value and p . This is optimal for all types of each bidder because, when $v_i > p$,

bidder i 's earns more in expectation by buying one object at p than by attempting to get both objects at a higher price.⁶

Clearly, the possibility of binding budget constraints modifies the equilibrium set. In particular, budget constraints limit competition both directly, because constrained bidders are unable to compete for both objects at all prices, and indirectly, as the restraint in competition may actually reduce the scope for collusion. The indirect effect is akin to the one that arises in repeated games, where collusion is easier to sustain when the payoffs of the noncooperative equilibria are lower. In a simultaneous ascending auction, a bidder is less likely to accept a collusive split of the objects if the price at which she expects to be able to win both objects is lower. The presence of budget constraints inevitably softens competition, and thus makes it more difficult to punish deviations from collusive behavior.

While important, the second effect is straightforward to understand. In this paper we ignore it, and focus on 'noncollusive' equilibria (formally defined in the next paragraph). This is because our goal is to study the impact that the possibility of binding budget constraints has on the level of efficiency sustainable in equilibrium. Thus from now on we restrict attention to equilibria in which the objects are allocated efficiently whenever it is common knowledge that the budget constraints are not binding. In particular, this implies that no bidder will ever reduce demand to one when the price is below h_L , since at $p < h_L$ it is common knowledge that both bidders can buy both objects. Thus, at any price $p < h_L$ both bidders will demand either two objects or none. Furthermore, if at $p > h_L$ both agents demand two objects, then it becomes common knowledge that both bidders are unconstrained, hence both bidders will bid up to their values on both objects.

Equilibria in which the objects are split before the price reaches h_L , or when it is clear that both bidders are unconstrained, have a collusive flavor. To rule out these possibilities, we define noncollusive equilibria as follows.

Definition. An equilibrium is *noncollusive* if:

1. whenever $\min\{v_1, v_2\} < h_L$, the bidder with the highest value wins both objects; and
2. if both bidders demand two objects at any $p > h_L$, then it becomes common knowledge that $h_1 = h_2 = h_H$ and the bidder with the highest value wins both objects.

It is worth pointing out that both conditions in the previous definition are necessary to achieve efficiency. In particular, the distortions from the first

⁶ See Bajari and Fox [2005] for an empirical study of complementarities and collusion in one of the FCC auctions.

best that we will identify in our analysis can only be larger in any equilibrium where the objects are split before the price reaches h_L .

In any noncollusive equilibrium the auction begins with bidder i bidding on both objects up to $\min\{v_i, h_L\}$. If the price reaches h_L , then any 'low-budget' type (v_i, h_L) can do no better than reducing demand to one and bidding on a single object up to v_i . If no bidder reduces her demand at $p = h_L$, then it becomes common knowledge that no bidder is constrained, and the auction continues until the bidder with the lower value stops bidding on both objects at her value.

Our first proposition establishes that in the absence of binding budget constraints – i.e., $\lambda = 0$ – there is a noncollusive equilibrium in which the outcome is efficient, i.e. both objects are always assigned to the bidder with the highest value. The proof consists in providing suitable out-of-equilibrium beliefs when one bidder reduces demand to one.

Proposition 1. If $\lambda = 0$, there exists an equilibrium in which bidder i buys both objects at price v_{-i} if $v_i > v_{-i}$, and no objects if $v_i < v_{-i}$.

In this equilibrium, each bidder bids on both objects up to her value. Thus a reduction of demand to one (rather than zero) is an out-of-equilibrium event. In order to support the outcome, we specify beliefs for which bidding up to her value on both objects is optimal for a bidder who sees her opponent conceding a single unit. The beliefs that we choose specify that if a bidder, say 1, releases one button at p then bidder 2 will think that $v_1 = p$, and thus will expect bidder 1 to give up the other object immediately. If this does not happen, bidder 2 will keep thinking that 1 will exit on the other object immediately.

The specification of out-of-equilibrium beliefs is crucial in order to support the efficient outcome. Whenever a bidder reduces her demand to one, the other bidder can terminate the auction at any time by also reducing demand to one. She will continue to bid on both objects because she believes that her opponent will quit immediately.

These beliefs however cannot be used when $\lambda < 0$, because then, in any noncollusive equilibrium, demand reduction to one at $p = h_L$ happens with positive probability on the equilibrium path: all low-budget types must give up one object because they can no longer afford to bid on both. In the next two sections we show that this has important consequences, and in fact can dramatically reduce the efficiency of the simultaneous ascending auction.

III. THE EFFECT OF PRIVATELY KNOWN BUDGET CONSTRAINTS

An outcome in our model can be described by a collection of functions (q_1, q_2, m_1, m_2) where

$$q_i : \Theta \rightarrow [0, 2] \quad \text{and} \quad m_i : \Theta \rightarrow \mathfrak{R}, i = 1, 2,$$

specify the number of objects $q_i(\theta)$ allocated to bidder i and the amount of money $m_i(\theta)$ that bidder i pays, for each profile $\theta = (\theta_1, \theta_2) \in \Theta$. We will write $q = (q_1, q_2)$ and $m = (m_1, m_2)$, so that an allocation is given by a pair (q, m) .

The presence of budget constraints makes full efficiency impossible: whenever $h_L = h_i < v_j < v_i$ bidder i will be unable to win both objects, as required by efficiency, because by Lemma 1 bidder j will not stop bidding on both objects until the price reaches v_j . Lemma 1 also implies that any bidder who wins both objects must pay twice her opponent's value. We record these properties that any noncollusive equilibrium outcome must satisfy in the next lemma.

Lemma 2. If (q, m) is the outcome of a noncollusive equilibrium then

$$(1) \quad q_i(v_i, h_i, v_{-i}, h_{-i}) = 1 \quad \text{if } h_i < v_{-i} < v_i, \quad i = 1, 2,$$

and

$$(2) \quad m_i(v_i, h_i, v_{-i}, h_{-i}) = \begin{cases} 2v_{-i}, & \text{if } q_i(v_i, v_{-i}, h_i, h_{-i}) = 2; \\ p_i(v_i, h_i, v_{-i}, h_{-i}), & \text{if } q_i(v_i, v_{-i}, h_i, h_{-i}) = 1; \\ 0, & \text{if } q_i(v_i, v_{-i}, h_i, h_{-i}) = 0; \end{cases}$$

for some integrable function $p_i: \Theta \rightarrow R_+$ such that $h_L \leq p_i(v_i, h_i, v_{-i}, h_{-i}) \leq v_i$.

Note that Lemma 2 places no restriction on the way the objects are allocated when the bidder with the higher value can afford to pay twice the opponent's value, i.e., when $v_j < \min\{v_i, h_i\}$. Any assignment function q^* which allocates the objects efficiently in all these cases, must satisfy⁷

$$(3) \quad q_i^*(\theta_i, \theta_j) = \begin{cases} 2, & \text{if } v_j < \min\{v_i, h_i\}, \\ 0, & \text{if } v_i < \min\{v_j, h_j\}, \\ 1, & \text{otherwise.} \end{cases}$$

In words, the assignment function q^* allocates one object to each bidder whenever the bidder with the highest value is unable to pay for both objects, and gives both objects to the bidder with the highest value otherwise. Thus, q^* is a 'constrained-efficient' assignment function, as it maximizes the social surplus subject to the constraint in (1). In Figure 1, each of the four diagrams corresponds to a realization of the bidders' budget levels; it has the bidders' values on the axes, and indicates the number of items that are awarded to bidder 1 (both objects are always sold). For example, the graph in the lower-right corner refers to the case where bidder 2 is the only constrained bidder ($h_2 < h_1$), and shows that bidder 1 is awarded both items whenever $v_2 < v_1$, one item when $h_2 < v_1 < v_2$, and no items when $v_1 < \min\{h_2, v_2\}$. When both

⁷ Ties, being zero probability events, are ignored.

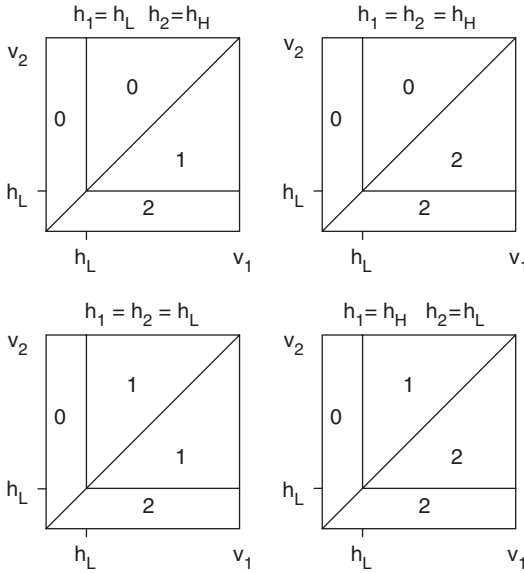


Figure 1
The assignment function q^* .

bidders are unconstrained the first-best outcome is achieved, as shown in the upper-right graph.

Is there a noncollusive equilibrium with outcome (q^*, m) , where m satisfies the restrictions in (2)? The next proposition establishes that the answer is no. Therefore, in addition to the direct effect captured by the constraint in (1), the possibility of binding budget constraints must induce other distortions from efficiency.

Proposition 2. For any $\lambda > 0$, any outcome (q^*, m) , where q^* is defined in (3) and m satisfies (2), violates incentive compatibility. Therefore q^* cannot be attained by any noncollusive equilibrium of the simultaneous ascending bid auction.

Note that our definition of noncollusive equilibria pins down the behavior of all low-budget types: types with $v_i < h_L$ bid on both objects up to v_i ; and types with $v_i > h_L$ bid on both objects up to $p = h_L$, and then on a single object up to v_i .

Thus the additional distortions from efficiency that, in light of Proposition 2, must be present in any noncollusive equilibrium can only be induced by the behavior of high budget types. Indeed, the proof of Proposition 2 consists in showing that the payment functions defined in (2) imply interim expected payments for all high-budget types with value above h_L that are lower than those implied by the constrained-efficient assignment function q^* and incentive compatibility.

The next proposition characterizes the set of all symmetric noncollusive equilibria. In particular it establishes that there always exists a set of positive measure of *high-budget* types with value above h_L that behave as if they were budget constrained: when the price arrives at h_L they reduce demand to one and bid on a single object up to their value. This ‘strategic’ demand reduction effect is responsible for the additional distortions implied by Proposition 2.

Proposition 3. In any symmetric noncollusive equilibrium, all types with value below h_L bid on both objects up to their value. Low-budget types with value above h_L bid on two objects up to h_L , and on one object up to their value. The strategy of high-budget types with value above h_L is characterized by a threshold $v^*(\lambda) \in (h_L, 1]$ and an optimal nondecreasing ‘stopping time’

$$s^* : [h_L, 1] \rightarrow [h_L, 1],$$

such that:

- all types with $v_i \in (h_L, v^*(\lambda))$ reduce demand to one at $p = h_L$, and bid up to their value; and
- all types with $v_i > v^*(\lambda)$ bid on both objects up to their value, unless the opponent reduces demand to one at h_L , in which case they bid on both objects up to an optimal ‘stopping time’ $s^*(v_i) < v_i$. Furthermore, there exists a threshold $v_a(\lambda) \in (h_L, v^*(\lambda))$ such that $s^*(v_i) = h_L$ for all $v_i \in [h_L, v_a(\lambda))$.

The bidders’ behavior on the equilibrium path can be described as follows. First, all types with value $v_i \leq h_L$ bid ‘straightforwardly,’ i.e., they bid on both objects until the price reaches their value and then reduce demand to zero. These types never face binding budget constraints, as they always have enough money to pay for both objects whenever their value is above the current price.

Consider now all types with value $v_i > h_L$. Low budget types must reduce demand to one when the price reaches h_L , since they can no longer afford to pay for both objects (this is the ‘direct’ demand reduction effect); they continue to bid on a single object up to their value v_i . High-budget types with $v_i > h_L$ behave as follows:

- types with value v_i close enough to h_L ‘pretend’ to be budget constrained: they reduce demand to one when the price reaches h_L (this is the ‘strategic’ demand reduction effect), and bid on one object up to their value;
- the remaining high-budget types—those with a sufficiently high value—behave as follows:
 - if the opponent reduces demand to one at $p = h_L$, they keep bidding on both objects up to a threshold $s^*(v_i)$, and then end the auction by releasing one button;

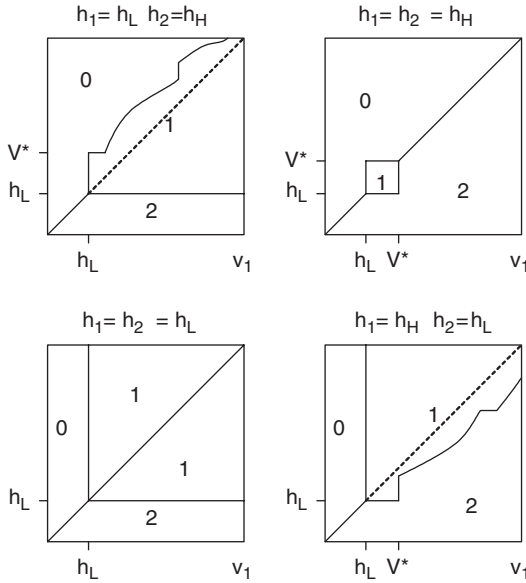


Figure 2
An Equilibrium Assignment Function.

- if the opponent does not reduce demand to one at $p = h_L$, they keep bidding on both objects up to v_i .

The structure of the set of high-budget types that pretend to be budget constrained depends on λ and F , but it always contains a non-empty interval $(h_L, v^*(\lambda))$.

Note that when each bidder is either budget constrained, or has value between h_L and $v^*(\lambda)$ and thus pretends to be budget constrained, the bidders reduce demand to one *simultaneously* when the price arrives at h_L . Thus the objects may end up being split even when both bidders are not budget constrained. Figure 2 illustrates an equilibrium assignment function.

Consider the upper-left panel, corresponding to the case where only bidder 1 has a low budget. The objects are split not only when $h_L < v_2 < v_1$ (in the triangle below the diagonal), due to ‘direct’ demand reduction by bidder 1, but also when bidder 2 has the higher value and strategically reduces her demand to one in two ways: i) by ‘pretending’ to be budget constrained, thus reducing demand simultaneously with bidder 1 at $p = h_L$ (when $h_L < v_2 < v^*$); and ii) by conceding one object at a price $s^*(v_2) < v_1$ (when v_2 is above v^* and below the irregular curve.)

The threshold $s^*(v_2)$ is the solution of a ‘monopsony’ problem: after bidder 1 has reduced demand to one at $p = h_L$ it is clear that she will bid on a single object up to v_1 . Thus bidder 2’s problem is simply to determine the optimal

threshold beyond which it is no longer optimal to bid on both objects. The trade-off between quantity and price the standard one from monopsony theory: thus it is not surprising that in general the optimal threshold is strictly below bidder 1's value, i.e., $s^*(v_1) < v_1$. Intuitively, this is because as the price approaches v_1 the expected surplus from trying to get both objects goes to zero, hence, it is better to buy a single object at $v_1 - \varepsilon > 0$, rather than letting the price reach v_1 .

Proposition 3 also establishes that there is always a nonempty interval $(h_L, v_a(\lambda))$ of high budget types with value near h_L that react their opponent's demand reduction at $p = h_L$ with an immediate reduction of their own demand, thus ending the auction at price h_L . That is, $s^*(v_1) = h_L$ for all $v_1 \in (h_L, v_a(\lambda))$. This in turn induces a set of high budget types, which includes a nonempty interval $(h_L, v^*(\lambda))$, to reduce demand to one at $p = h_L$, without waiting to see if the opponent is doing the same. Therefore, in the upper right panel, where both bidders have the high budget, there is a square region in which the objects are split.

We close this section with the observation that the probability of observing demand reduction when the price reaches h_L is always higher than λ : not only all low-budget types, but also some of the high-budget types reduce demand. This suggests that, depending on the structure of the set of high-budget types who reduce demand to one at h_L , the probability of having the objects split may remain significant as λ goes to zero. In the next section we show that, under mild conditions on the value distribution F , the probability of observing strategic demand reduction remains bounded away from zero as λ goes to zero.

IV. VANISHING PROBABILITY OF BUDGET CONSTRAINTS

Let S_j denote the event that bidder $j \neq i$ reduces demand to 1 when the price reaches h_L , and let $\Phi(\cdot | S_j)$ denote the c.d.f. on v_j conditional on S_j . Furthermore, let M_λ be the set of types who reduce demand to one when the price reaches h_L , and let $\mu(\lambda) \equiv \int_{M_\lambda} dG(x)$ be the measure of the set M_λ . The c.d.f. that represents bidder i 's beliefs on her opponent's value, conditional on j reducing demand to one at h_L can be written as

$$\begin{aligned} \Phi_\lambda(y | S_j) &= \frac{\lambda}{\lambda + (1 - \lambda)\mu(\lambda)} G(y) + \frac{(1 - \lambda)\mu(\lambda)}{\lambda + (1 - \lambda)\mu(\lambda)} G(y | M_\lambda) \\ &= \frac{1}{1 + (1 - \lambda)\frac{\mu(\lambda)}{\lambda}} G(y) + \frac{(1 - \lambda)\frac{\mu(\lambda)}{\lambda}}{1 + (1 - \lambda)\frac{\mu(\lambda)}{\lambda}} G(y | M_\lambda). \end{aligned}$$

What happens when $\lambda \downarrow 0$? In particular, can we get efficiency in the limit? In order to have efficiency in the limit, one requirement is that a bidder facing an opponent who has reduced demand to one is willing to bid on both objects up to her value. This essentially requires that $\Phi(y | S_j)$ puts probability one on

$v_j = h_L$, i.e. $\Phi(y|S_j) = 1_{[y \geq h_L]}(y)$. Thus, we have to evaluate what happens to $\lim_{\lambda \downarrow 0} \Phi_\lambda(y|S_j)$. This in turn depends on the value of

$$\lim_{\lambda \downarrow 0} \frac{\mu(\lambda)}{\lambda}.$$

If the limit is finite, then $\lim_{\lambda \downarrow 0} \Phi_\lambda(y|S_j)$ will be a strictly increasing function, and thus a strictly positive measure of types will accept splitting the objects and also will propose the split; in this case efficiency is not achieved as λ goes to zero. Thus, in order to approach efficiency in the limit we need

$$(4) \quad \lim_{\lambda \downarrow 0} \frac{\mu(\lambda)}{\lambda} = +\infty$$

and

$$(5) \quad \lim_{\lambda \downarrow 0} G(y|M_\lambda) = 1_{[y \geq h_L]}(y).$$

In that case the conditional distribution converges exactly to the one needed to support the efficient outcome, and the inefficiency will disappear as λ goes to zero. Notice that condition (5) can occur only if $\lim_{\lambda \downarrow 0} \mu(\lambda) = 0$.

We will prove that there is a class of distributions for which the equalities do *not* hold. Thus, for such distributions the demand reduction effect remains bounded away from zero as λ goes to zero. In other words, for such distributions even an arbitrarily small probability of having binding budget constraints has first-order consequences on efficiency. The result is contained in the next proposition.

Proposition 4. Suppose that the density f is non-decreasing on the interval $[h_L, v']$, for some $v' \in (h_L, 1]$. Then:

1. $v^*(\lambda) \geq v'$ for each $\lambda > 0$, hence $\lim_{\lambda \downarrow 0} v^*(\lambda) \geq v' > h_L$;
2. if $v' = 1$, then for each $\lambda > 0$ there exists a unique noncollusive equilibrium. In this equilibrium all high-budget types reduce demand to one when the price reaches h_L , i.e., $M_\lambda = [h_L, 1]$.

Proposition 4 implies that, when f is non-decreasing on an interval $[h_L, v']$ the inefficiency will not tend to zero as λ goes to zero. In any noncollusive equilibrium, it must be the case that types (v_i, h_L) with $v_i \in (h_L, v')$ reduce their demand to one at $p = h_L$. Thus, no matter how small λ is, M_λ will contain the interval (h_L, v') .

Part 2 of the proposition additionally states that when f is nondecreasing on the whole interval $[h_L, 1]$ that there exists a unique non-collusive equilibrium for each $\lambda > 0$, and in that equilibrium $v^*(\lambda) = 1$. The equilibrium has the remarkable property that it does not depend on λ , the probability that any bidder is budget-constrained: for any $\lambda > 0$, in the

unique noncollusive equilibrium all the bidders mimic the low-budget bidders when the price reaches h_L . This is a striking example of strategic demand reduction. As soon as it becomes possible to mimic a low-budget bidder (that is, for any $\lambda > 0$), all bidders will act as if they were budget constrained.

Since for $\lambda = 0$ there is an equilibrium in which the objects are allocated efficiently, Proposition 4 implies that the set of noncollusive equilibria changes dramatically as soon as $\lambda > 0$. Intuitively, this is due to the fact that, as λ becomes positive, a new set of types (the low-budget ones) is introduced into the model, hence in equilibrium an additional family of incentive compatibility constraints must be satisfied. Given the restrictions that the auction rules and the requirement that bidders use undominated strategies put on the bidders' payments, these constraints cannot be satisfied by any outcome in which the objects are assigned as in the assignment function q^* defined in (3).

In particular, for any $\lambda > 0$ there is an interval $[h_L, v^{*,1}]$ of values such that all types (v_i, h_i) with $v_i \in [h_L, v^{*,1}]$ stop immediately as soon as the opponent starts bidding defensively, i.e., pursuing only one object up to her value. This induces another set of types, with values in an interval $[h_L, v^{-1}]$ to behave as if their budget were h_L , that is to start bidding defensively at h_L . At this point the probability distribution on an opponent who starts bidding defensively at h_L changes, and we can compute another interval $[h_L, v^{*,2}]$ of values such that all types (v_i, h_i) with $v_i \in [h_L, v^{*,2}]$ stop immediately as soon as the opponent starts bidding defensively. Next we determine a new interval $[h_L, v^{-2}]$, and so on. Continuing this process, we converge to the values threshold $v^*(\lambda)$ and $v_a(\lambda)$ such that all high-budget types with value in the interval $[h_L, v^*(\lambda)]$ prefer to behave as low-budget types.

In other words, for any $\lambda > 0$, the presence of low-budget types triggers a 'contagion effect': some high-budget types will always mimic the behavior of low-budget types, and this in turn induces more high-budget types to do the same and so on. When $\lambda = 0$ the high-budget types cannot 'hide' behind low-budget types, i.e. the incentive compatibility constraints due to the possibility of binding budget constraints disappear, and there is no 'starting point' on which to build the contagion effect.

V. DISCUSSION

In this section we discuss the generality and significance of our results.

V(i). *Multiple Budget Levels*

Our analysis has been restricted to the case in which only two budget levels are possible. All results can be immediately extended to the case in which

there is a finite set of possible budget levels $\{h_1, h_2, \dots, h_K\}$, $\lambda_k > 0$ being the probability that the budget of a bidder is h_k .

The case of a continuum of budgets is more complicated, but the demand reduction effect will remain. Consider the following distribution on budgets. The value h_i is higher than one with probability $(1 - \lambda)$, while with probability λ it is drawn from a distribution with c.d.f. $Z(h_i)$ (the model that we have studied in the previous sections is the special case where $Z(h_i) = 1_{[h_L, 1]}(h_i)$). In any noncollusive equilibrium, reducing demand to one when the price reaches $p = h_L$ must be observed on the equilibrium path. Facing an opponent who has reduced demand to one, a bidder of type (v_i, h_i) will choose an optimal stopping time solving

$$\max_{s \in [h_L, h_i]} \int_{h_L}^s 2(v_i - y) d\Phi(y|S_j) + (v_i - s)[1 - \Phi(s|S_j)].$$

The ‘monopsony effect’ will still be present; that is, a bidder facing an opponent who has conceded one object will not push the price up to her value, even if this is budget-feasible. It is therefore still true that, by conceding one object at $p = h_L$, a bidder can induce her opponent to compete less aggressively. This in turn implies that in any equilibrium there will be types who reduce their demand to one before the price reaches their value in order to soften competition. The basic mechanism for demand reduction therefore works also in this case, and it is not difficult to work out examples in which as λ goes to zero, the demand reduction effect does not disappear. For example, when f is nondecreasing, having all types reducing demand to one at h_L is still a noncollusive equilibrium. In fact, exactly the same strategies and beliefs used in the proof of Proposition 4 work.⁸ Furthermore, it is possible to develop examples in which the distribution $Z(h_i)$ is ‘close enough’ to $1_{[h_L, 1]}(h_i)$ so that, no matter what the value of λ is, all types reduce their demand to one at h_L .

V(ii). *Can the Inefficiency be Eliminated?*

The strategic demand reduction effect hinges on the fact that, in the auction format we consider, the prices of *both* objects keep increasing as long as aggregate demand is greater than aggregate supply. In fact, the demand reduction effect that we find here also appears in other auction formats, under suitable notions of ‘noncollusive’ behavior. For example, suppose that, rather than with a clock auction, the objects are sold with a simultaneous ascending bid auction in which the bidders are allowed to

⁸ The only modification in the proof is that we need to observe that the expected utility of a type (v_i, h_i) when not reducing demand and selecting an optimal stopping type is nonincreasing in h_i . Thus, if the deviation is not profitable for (v_i, h_H) , it will not be profitable for (v_i, h_i) with $h_i < h_H$.

bid freely on both objects (i.e., there is no activity rule). If a bidder who is trying to buy a single object always bids on an object with the current lowest price, then demand reduction effects similar to the one that we have found in our auction format will appear. (Bidding on the object with the lower price is a reasonable notion of ‘noncollusive’ behavior in this case, as bidding on the other object must involve some form of signalling or retaliation.)

More generally, the demand reduction effect appears whenever a bidder faces an opponent who is trying to buy a single object and in doing so is causing an increase in the prices of both. Whenever this happens, the ‘monopsony effect’ will appear, and the bidder who is trying to buy both objects will stop before the prices reach her value. This in turn creates incentives for further demand reduction, as there will be types who reduce their demand to one, even if they are not budget constrained, in order to induce their opponent to bid less aggressively.

The demand reduction effect can generate significant efficiency losses. In proposition 4 we have shown that, for densities which are increasing on the interval $[h_L, 1]$, in the unique noncollusive equilibrium both objects are split whenever $\min\{v_1, v_2\} \geq h_L$. Thus, the welfare loss as λ goes to zero is

$$E[|v_2 - v_1| : \min\{v_1, v_2\} \geq h_L] \cdot \Pr[\min\{v_1, v_2\} \geq h_L].$$

What can an auction designer do to correct this? A simple measure is to introduce a reservation price, or to ask the bidders to post a bond, in order to exclude low budget bidders.⁹ Without budget constraints, measures that exclude certain types of bidders unambiguously reduce the social surplus because they prevent potential gains from trade from being realized. With potentially binding budget constraints however, there are distributions for which, even in noncollusive equilibria, there is a high probability that bidders end up buying one object each for a low price, even when they are not budget-constrained. Splitting the objects lowers the social surplus, since efficiency requires that both objects be assigned to the bidder with the higher valuation. When budget constrained bidders are excluded from the auction it becomes common knowledge that all active bidders are unconstrained. Therefore, if the bidders are not colluding, each object always ends up in the hands of the bidder with the highest value. If the probability of binding budget constraints is sufficiently small, the expected gain in social surplus due to the better allocation of the objects is larger than the expected loss due to the exclusion of budget constrained types.¹⁰

⁹ Cramton and Schwartz [2000] have suggested that reservation prices may be used to upset collusion in multi-unit auctions. Their paper contains an example with complete information. It is easy to construct examples where reservation prices can actually increase welfare in noncollusive equilibria when the possibility of binding budget constraints is admitted.

¹⁰ A word of caution is in order here. The auction has multiple equilibria, and some of them are highly collusive. Restricting the participation of low-budget bidders makes sense only if we believe that high-budget bidders will bid according to noncollusive strategies.

If the mechanism designer is willing to change the auction format, then the demand reduction effect can be eliminated using an 'object-specific' clock auction can generate a higher social surplus if budget constraints are present. In such an auction, once a bidder gives up one object, say object 1, the price is fixed and the object is allocated to the other bidder. The two bidders may keep competing on the other object, say object 2, but at that point the increase in the price of object 2 does not change the price to be paid for object 1. Thus, in the 'object-specific' button auction there are no incentives to reduce demand; notice however that any format in which *both* prices increase as long as aggregate demand is higher than aggregate supply is not immune from the sort of demand reduction induced by the presence of budget constraints that we have analyzed in this paper.

VI. CONCLUSIONS

We have explored the impact that the possibility of binding budget constraints may have on auctions with multiple objects. It is clear that budget constraints reduce the level of competition because the bidders have a lower ability to pay. We have shown that competition is further reduced due to strategic reasons. The strategic effect is significant. For a large class of distributions, even if the probability of having binding budget constraints is arbitrarily small, a set of positive measure of high-budget types will behave as if they were budget constrained. Therefore the objects will be inefficiently split with positive probability, even when no bidder is budget constrained. This outcome appears to be collusive, even if the bidders use noncollusive strategies. The problem is created by the fact that the presence of low-budget bidders introduces additional incentive compatibility constraints. One important implication is that measures leading to the exclusion of budget-constrained bidders from the auction can be welfare enhancing (as well as revenue enhancing), since they stimulate competition and favor a more efficient allocation of the objects.

More generally, the demand reduction effect that we have identified will be present in any auction mechanism in which the prices of *all* the objects increase as long as aggregate demand is greater than aggregate supply. Our results suggest that ignoring the possibility of binding budget constraints in auction design can generate significant deviations from efficient outcomes. If the auction designer is worried that the possibility of binding budget constraint may generate the kind of inefficiencies described in this paper, then possible remedies include reservation prices that exclude low-budget bidders or changing the rules of the auction so that once a bidder gives up an object, the price of that object stops increasing.

APPENDIX

Proof of Lemma 1. Consider bidder 1. If $p_t < v_1$, and bidder 1 reduces her demand to zero, the auction ends. In this case, bidder 1's payoff is zero, unless bidder 2 also reduces her demand at p_t in which case bidder 1 earns $v_1 - p_t$. Consider now the alternative strategy of reducing demand to one at p_t , and to zero when the price reaches v_1 . Bidder 1's payoff in this case depends on bidder 2's behavior as follows: i) if 2 also reduces her demand at p_t , the outcome is the same as the one induced by the original strategy (of reducing demand to zero at p_t); ii) if bidder 2 reduces her demand at any price $p_{t'} \in (p_t, v_1)$, then bidder 2 earns $v_1 - p_{t'}$ which is positive; iii) if bidder 2 does not reduce demand before the price reaches v_1 , then bidder 1 earns zero, as under the original strategy. We conclude that the alternative strategy does strictly better in some cases and at least as well in all other cases, for any $p_t < v_1$.

Keeping a positive demand up to $p_t < v_1$ yields negative surplus if bidder 2 reduces her demand to zero at any price $p_{t'} \in (v_1, p_t)$, while reducing demand to zero when the price reaches v_1 yields a higher expected surplus. ■

Proof of Proposition 1. We specify the strategy and the beliefs of bidder i as follows:

- Keep two buttons pushed if $p < v_i$, and release both buttons if $p \geq v_i$, independently of whether your opponent is pushing one button or two.
- If your opponent is pushing both buttons at price p then the conditional distribution on her type is $F(v_{-i} | v_{-i} > p)$. If the opponent is keeping one button pushed at price p then the conditional distribution is

$$\Phi(v_{-i}|p) = \begin{cases} 0 & \text{if } v_j < p \\ 1 & \text{if } v_j \geq p \end{cases}$$

Releasing just one button is an out-of-equilibrium strategy. Under the described beliefs it is optimal for bidder i to keep both buttons pushed when the opponent has released just one button, provided that the price is below v_i . Furthermore, it cannot be optimal to release just one button. If bidder i does that, she still gets zero objects whenever $v_{-i} > v_i$, and gets only one object (rather than the two that would be obtained under the equilibrium strategy) at price v_{-i} when $v_{-i} < v_i$. ■

Proof of Lemma 2. Consider any type profile such that $h_i < v_{-i} < v_i$. By the auction rules, it must be the case that $q_1(\theta_1, \theta_2) + q_2(\theta_2, \theta_1) = 2$. If $q_i(\theta_i, \theta_{-i}) = 2$ then the objects must have been sold at $p \leq h_i$, because bidder i cannot bid on both objects when the price is higher than h_i . Thus $p \leq h_i < v_{-i}$. But then bidder $j \neq i$ must have stopped bidding on both objects at a price strictly lower than v_j , contradicting point 1 in Lemma 1. If $q_{-i}(\theta_i, \theta_{-i}) = 2$, the sale price p cannot exceed v_{-i} , by point 2 in Lemma 1. But $p \leq v_{-i} < v_i$ implies that i stops bidding on both objects before the price reaches v_i , contradicting point 1 in Lemma 1. We conclude that the objects have to be split whenever $h_i < v_{-i} < v_i$.

Lemma 1 directly implies that the price must be v_{-i} when i gets both objects, and the auction rules imply that if a bidder gets zero objects, her payment must be zero.

The inequality $p_i(\theta_i, \theta_{-i}) \geq h_L$ follows because when the price is below h_L it is common knowledge that the bidders are not constrained, and therefore in a noncollusive equilibrium the objects are not split. The inequality $p_i(\theta_i, \theta_{-i}) \leq v_i$ follows directly from Lemma 1. Finally, since (q, m) is an equilibrium outcome, by the revelation principle the interim functions $Q_i(\theta_i) \equiv E_{\theta_{-i}}[q_i(\theta_i, \theta_{-i})]$ and $M_i(\theta_i) \equiv E_{v_{-i}, h_{-i}}[m_i(\theta_i, \theta_{-i})]$ must satisfy the incentive compatibility constraints

$$(6) \quad v_i Q_i(\theta_i) - M_i(\theta_i) \geq v_i Q_i(\theta'_i) - M_i(\theta'_i) \quad \forall \theta_i, \theta'_i \in \Theta_i.$$

In particular, M_i must be well defined, hence p_i must be integrable. ■

Proof of Proposition 2. Suppose that (q^*, m) is incentive-compatible. First observe that any type of bidder i with $v_i = 0$ never wins, and this implies $M_i(0, h_L) = M_i(0, h_H) = 0$. Then, by standard mechanism design arguments, the *interim* expected payment of any high-budget type (v_i, h_H) of bidder i is

$$(7) \quad M_i(v_i, h_H) = v_i Q^*(v_i, h_H) - \int_0^{v_i} Q^*(y, h_H) dy, \quad \forall v_i \in [0, 1],$$

where

$$(8) \quad Q^*(v_i, h_H) \equiv E_{\theta_{-i}}[q_i^*(v_i, h_H, \theta_{-i})] \\ = \begin{cases} 2F(v_i), & \text{if } v_i \leq h_L, \\ 2F(v_i) + \lambda[1 - F(v_i)], & \text{if } v_i > h_L. \end{cases}$$

In fact, when $v_i \leq h_L$ bidder i wins two objects if $v_{-i} < v_i$, and zero otherwise, thus yielding an expected quantity of $2F(v_i)$. When $v_i > h_L$, bidder i (who is not budget constrained) wins both objects if $v_{-i} < v_i$, which happens with probability $2F(v_i)$, and one object if $v_{-i} > v_i$ and her opponent is budget constrained, which happens with probability $\lambda[1 - F(v_i)]$. Substituting (8) into (7) we have that, for each $v_i \in (h_L, 1]$,

$$(9) \quad M_i(v_i, h_H) = [2F(v_i) + \lambda[1 - F(v_i)]]v_i - \int_0^{v_i} 2F(y)dy - \lambda \int_{h_L}^{v_i} [1 - F(y)]dy \\ = 2 \left[v_i F(v_i) - \int_0^{v_i} F(y)dy \right] + \lambda \left[[1 - F(v_i)]v_i - \int_{h_L}^{v_i} [1 - F(y)]dy \right] \\ = 2 \int_0^{v_i} y dF(y) + \lambda \left[[1 - F(h_L)]h_L - \int_{h_L}^{v_i} y dF(y) \right],$$

where the last equality follows after integrating by parts both terms in the second line.

Since in any noncollusive equilibrium m_i must satisfy the restrictions in (2), we also have

$$(10) \quad M_i(v_i, h_H) = 2 \int_0^{v_i} y dF(y) + \lambda \int_{v_i}^1 p_i(v_i, y, h_H, h_L) dF(y) \quad \forall v_i \in [h_L, 1].$$

for some integrable function $p_i(\cdot)$ such that $p_i(\cdot) \geq h_L$. Combining (9) and (10) and simplifying yields

$$\lambda \left([1 - F(h_L)]h_L - \int_{h_L}^{v_i} y dF(y) \right) = \lambda \int_{v_i}^1 p_i(v_i, y, h_H, h_L) dF(y).$$

By Lemma 2, we have $p_i(v_i, y, h_H, h_L) \geq h_L$. Thus

$$\lambda \left([1 - F(h_L)]h_L - \int_{h_L}^{v_i} y dF(y) \right) \geq \lambda [1 - F(v_i)]h_L,$$

or equivalently

$$\lambda \left([F(v_i) - F(h_L)]h_L - \int_{h_L}^{v_i} y dF(y) \right) = \lambda \int_{h_L}^{v_i} (h_L - y) dF(y) \geq 0,$$

which is impossible if $\lambda > 0$. ■

Proof of Proposition 3.

The proof consists of two lemmas (3 and 4) and a final part.

Lemma 3. In any noncollusive equilibrium, the strategy of almost all high-budget types of bidder i with $v_i \geq h_L$, conditional on observing the opponent reducing demand to one object at h_L , is characterized by an optimal ‘stopping time’

$$s_i^* : [h_L, 1] \rightarrow [h_L, 1]$$

with $s_i^*(v_i) < v_i$. Furthermore, $s_i^*(\cdot)$ is nondecreasing.

Proof of Lemma 3. Suppose that the price arrives at h_L and bidder $j \neq i$ reduces demand to one. Bidder i must then decide, at any price $p < v_i$, whether to end the auction and buy a single object at p by reducing her own demand to one, or keep bidding on both objects. Thus her strategy must prescribe an optimal stopping time

$$s_i^*(v_i) \in \arg \max_{s \in [h_L, v_i]} U_i(s, v_i | S_j),$$

where

$$U_i(s, v_i | S_j) \equiv 2 \int_{h_L}^s (v_i - y) d\Phi(y | S_j) + (v_i - s) [1 - \Phi(s | S_j)].$$

The proof that $s^*(v_i) < v_i$ for almost all $v_i \in [h_L, 1]$ is in two steps. First we show that, for almost all $v_i \in [h_L, 1]$, if $\Phi(v_i|S_j) < 1$, then $s^*(v_i) < v_i$. After that, we will show that in any noncollusive equilibrium, $\Phi(v_i|S_j) < 1$ for all $v_i \in [h_L, 1]$.

For any $s < v_i$, we have

$$\begin{aligned} U_i(v_i, v_i|S_j) - U_i(s, v_i|S_j) &= 2 \int_s^{v_i} (v_i - y) d\Phi(y|S_j) - (v_i - s)[1 - \Phi(s|S_j)] \\ &\leq 2 \int_s^{v_i} (v_i - s) d\Phi(y|S_j) - (v_i - s)[1 - \Phi(s|S_j)] \\ &= (v_i - s) \{ [\Phi(v_i|S_j) - \Phi(s|S_j)] - [1 - \Phi(v_i|S_j)] \}. \end{aligned}$$

If $\Phi(v_i|S_j) < 1$, then $1 - \Phi(v_i|S_j) > 0$. If Φ is continuous at v_i , then there exists δ small enough such that

$$|\Phi(v_i|S_j) - \Phi(s|S_j)| < 1 - \Phi(v_i|S_j),$$

hence $U_i(v_i, v_i|S_j) < U_i(s, v_i|S_j)$ for all $s \in (v_i - \delta, v_i)$. We conclude that, if v_i is not a point of discontinuity for Φ , the optimal stopping time must be $s_i^*(v_i) < v_i$. Finally, since Φ is increasing, the set of its points of discontinuity is at most countable, hence of measure zero. We conclude that $s_i^*(v_i) < v_i$ almost everywhere, whenever $\Phi(v_i|S_j) < 1$.

We now show that in every noncollusive equilibrium it must be the case that $\Phi(v_i|S_j) < 1$ when $v_i < 1$. In a noncollusive equilibrium, there is no demand reduction to one at any price $p < h_L$. Once the price reaches h_L , all low-budget types with value above h_L reduce demand to one. Thus demand reduction to one at $p = h_L$ is a positive probability event on the equilibrium path. The conditional distribution $\Phi(v|S_j)$ can be computed as follows. First, all low-budget types of bidder j with $v_j \in (h_L, 1]$ must reduce their demand. Therefore

$$\Pr[v_j \leq y, S_j | h_j = h_L] = G(y) y \in [h_L, 1].$$

Also, let $M \subset [h_L, 1]$ denote the (possibly empty) set of high-budget types who reduce demand to 1 at $p = h_L$ (M may be empty). We have

$$\Pr[v_j \leq y, S_j | h_j = h_H] = \int_{[h_L, y] \cap M} dG(x).$$

Finally, we have

$$\begin{aligned} \Pr[S_j] &= \Pr[S_j | h_j = h_L] \Pr[h_j = h_L] + \Pr[S_j | h_j = h_H] \Pr[h_j = h_H] \\ &= \lambda + (1 - \lambda) \int_{[h_L, y] \cap M} dG(x) > 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \Phi(y|S_j) &= \frac{\Pr(v_j \leq y, S_j | h_i = h_L) \Pr(h_i = h_L) + \Pr(v_j \leq y, S_j | h_i = h_H) \Pr(h_i = h_H)}{\Pr(S_j)} \\ &= \frac{\lambda G(y) + (1 - \lambda) \int_{[h_L, y] \cap M} dG(x)}{\Pr(S_j)}. \end{aligned}$$

Since G is strictly increasing, $\Phi(\cdot | S_j)$ is strictly increasing on the interval $(h_L, 1)$. Therefore, for all $v_j < 1$, we have $\Phi(v_j | S_j) < \Phi(1 | S_j) = 1$.

Finally, since $\Phi(\cdot | S_j)$ is strictly increasing, the function $U_i(s, v_i | S_j)$ satisfies strict increasing differences in $(v_i; s)$. It follows¹¹ that any selection $s^*(v_i)$ is increasing. ■

The next lemma establishes that in every noncollusive equilibrium there is always an interval of positive measure of high budget types who react to a reduction in demand to one by the opponent at $p = h_L$ with an immediate reduction to one of their demand, thus ending the auction at price h_L .

Lemma 4. For any $\lambda > 0$, in any symmetric noncollusive equilibrium, there exists a threshold $v_a(\lambda) \in (h_L, 1]$ such that all high-budget types have stopping time $s_i^*(v_i) = h_L$ if $v_i < v_a(\lambda)$, and $s^*(v_i) > h_L$ if $v_i > v_a(\lambda)$.

Proof of Lemma 4. Fix $\lambda > 0$, and let M_λ denote the (possibly empty) set of high-budget types that reduce their demand to one when the price reaches h_L . The density of v_j , conditional on S_j , has support $[h_L, 1]$ and is equal to

$$\phi(y|S_j) = \frac{g(y)}{\lambda + (1 - \lambda) \int_{M_\lambda} dG(x)} [\lambda + (1 - \lambda) \mathbf{1}_{M_\lambda}(y)].$$

Since f is bounded, i.e., there exists k such that $f(v_i) \leq k$ for all $v_i \in [0, 1]$, and $g(y) = \frac{f(y)}{1 - F(h_L)}$ for $y \geq h_L$, we have $\phi(y|S_j) \leq \frac{k}{(1 - F(h_L))\lambda} \equiv k_\lambda$ for each $y \in (h_L, 1]$, which in turn implies $\Phi(y|S_j) \leq k_\lambda(y - h_L)$. Define $\tilde{v}_\lambda := \min\left\{h_L + \frac{1}{2k_\lambda}, 1\right\}$. Clearly, $\tilde{v}_\lambda > h_L$. Whenever $h_L < s < v_i < \tilde{v}_\lambda$, we have

$$\begin{aligned} U_i(s, v_i | S_j) - U_i(h_L, v_i | S_j) &= 2 \int_{h_L}^s (v_i - y) d\Phi(y|S_j) \\ &+ (v_i - s)[1 - \Phi(s|S_j)] - (v_i - h_L) \leq (v_i - h_L)\Phi(s|S_j) \\ &- (s - h_L)[1 - \Phi(s|S_j)], \end{aligned}$$

¹¹ See e.g. Milgrom and Shannon [1994].

hence

$$\begin{aligned} \frac{U_i(s, v_i|S_j) - U_i(h_L, v_i|S_j)}{s - h_L} &\leq \frac{\Phi(s|S_j)}{s - h_L} (v_i - h_L) - [1 - \Phi(s|S_j)] \\ &\leq k_\lambda(v_i - h_L) - 1 + k_\lambda(s - h_L) < 2k_\lambda(\tilde{v}_\lambda - h_L) - 1 \leq 0. \end{aligned}$$

Thus $U_i(s, v_i|S_j) < U_i(h_L, v_i|S_j)$ for each $s > h_L$. This implies

$$\{h_L\} = \arg \max_{s \in [h, v_i]} U_i(h_L, v_i|S_j)$$

for all $v_i \in (h_L, \tilde{v}_\lambda)$. We conclude that there exists a non-empty interval (h_L, \tilde{v}_λ) such that types (v_i, h_H) with $v_i \in (h_L, \tilde{v}_\lambda)$ have $s^*(v_i) = h_L$. This in turn implies

$$v_a(\lambda) = \sup\{v_i | s^*(v_i) = h_L\} > h_L.$$

Finally, since s^* is nondecreasing it follows that for $v_i > v_a(\lambda)$ we have $s^*(v_i) > h_L$. ■

The final step consists in showing that there exists a non-empty interval of high-budget types who reduce their demand to one as soon as the price reaches h_L ; that is, they reduce demand immediately, rather than as a reaction their opponent's reduction.

This is done by first proving that there is a set M_λ of positive measure such that all high-budget types of bidder i with value in M_λ reduce their demand to one when the price reaches h_L , and then showing that M_λ contains an interval $[h_L, v^*(\lambda)]$.

Suppose that only low-budget types reduce demand to one at h_L . In this case the demand reduction event S_j simply indicates that $h_j = h_L$ and provides no information on the value v_j . Therefore bidder i 's belief on v_j when the price arrives at h_L is $G(y) = F[y | v \geq h_L]$, independent of S_j .

Let $v_a(\lambda) > h_L$ be the value defined in Lemma 4: all high-budget types with value $v_i \in [h_L, v_a(\lambda)]$ stop immediately after seeing that the opponent has reduced demand to one at h_L . We now show that there is a non-empty interval (h_L, v^*) such that each high-budget type with $v_i \in [h_L, v^*]$ is strictly better off reducing demand to one at h_L . This is a contradiction, since we have assumed that no high-budget type reduces demand to 1 at h_L .

Consider a type (v_i, h_H) with $v_i \in (h_L, v_a(\lambda))$. For this type it is optimal to reduce demand to one whenever the opponent reduces demand to one at h_L . However, this type is not supposed to reduce demand to one unless she sees the opponent doing so. The expected surplus from reducing demand to one, conditional on the price having reached h_L , is

$$\begin{aligned} U_i^R(v_i, h_H) &= [\lambda + (1 - \lambda)G(v_a(\lambda))](v_i - h_L) \\ &\quad + (1 - \lambda) \int_{v_a(\lambda)}^1 \max\{v_i - s_i^*(y), 0\} dG(y), \end{aligned}$$

while not reducing demand to one and stopping immediately if the opponent reduces demand yields

$$U_i^N(v_i, h_H) = \lambda(v_i - h_L) + (1 - \lambda) \int_{h_L}^{v_i} 2(v_i - y)dG(y).$$

Now observe that

$$\begin{aligned} U_i^R(v_i, h_H) - U_i^N(v_i, h_H) &= (1 - \lambda) \left[G(v_a(\lambda))(v_i - h_L) \right. \\ &\quad \left. + \int_{v_a(\lambda)}^1 \max\{v_i - s_*(y), 0\}dG(y) - 2 \int_{h_L}^{v_i} [v_i - y]dG(y) \right] \\ &\geq (1 - \lambda) \left[G(v_a(\lambda))(v_i - h_L) - 2 \int_{h_L}^{v_i} [v_i - y]dG(y) \right] \equiv K(v_i). \end{aligned}$$

Since $K(h_L) = 0$, and

$$\left. \frac{dK}{dv_i} \right|_{v_i=h_L} = (1 - \lambda)G(v_a(\lambda)) > 0,$$

there exists an interval (h_L, v^*) such that for $v_i \in (h_L, v^*)$ we have $U_i^R(v_i, h_H) > U_i^N(v_i, h_H)$, a contradiction. We conclude that, in every noncollusive equilibrium, the set M_λ has strictly positive measure.

Now we prove that M_λ contains an interval $[h_L, v^*(\lambda)]$. Since the set M_λ is non-empty, the probability of observing the opponent reducing demand to one at h_L is

$$\Pr[S_j] = \lambda + (1 - \lambda) \int_{M_\lambda} dG(y).$$

Similarly, the probability that the opponent will not reduce demand to 1 is

$$1 - \Pr[S_j] = (1 - \lambda) \int_{N_\lambda} dG(y),$$

where $N_\lambda = [h_L, 1] \setminus M_\lambda$. If $\int_{N_\lambda \cap [h_L, v^*]} dG(y) = 0$ for some $v^* > h_L$ then the claim is proved. Therefore, from this point on we assume $\int_{N_\lambda \cap [h_L, v^*]} dG(y) > 0$ for each $v^* \in (h_L, 1)$. The density of v_j conditional on S_j is

$$\phi(y|S_j) = \frac{g(y)}{\lambda + (1 - \lambda) \int_{M_\lambda} dG(x)} [\lambda + (1 - \lambda)1_{M_\lambda}(y)].$$

The density of v_j conditional on S_j not happening is

$$\phi(y|\neg S_j) = \frac{(1 - \lambda)g(y)1_{N_\lambda}(y)}{(1 - \lambda) \int_{N_\lambda} dG(y)} = \frac{g(y)}{\int_{N_\lambda} dG(y)} 1_{N_\lambda}(y).$$

If demand is reduced to one, the expected utility is

$$U_i^R(v_i, h_H) = \Pr[S_j](v_i - h_L) + (1 - \Pr[S_j])\underline{U}_i(v_i),$$

where

$$\underline{U}_i(v_1) \equiv \int_{h_L}^1 \max\{v_i - s_j^*(y), 0\} d\Phi(y|\neg S_j).$$

If demand is not reduced, the expected utility is

$$U_i^N(v_i, h_H) = \Pr[S_j]\bar{U}_i(v_i) + (1 - \Pr[S_j])2 \int_{h_L}^{v_1} (v_i - y) d\Phi(y|\neg S_j),$$

where

$$\bar{U}_i(v_1) \equiv \int_{h_L}^{s_i^*(v_i)} 2(v_i - y) d\Phi(y|S_j) + [1 - \Phi(s_i^*(v_i)|S_j)](v_i - s_i^*(v_i)).$$

Pick v_i such that $h_L < v_i < v_a(\lambda)$, where $v_a(\lambda)$ was defined in Lemma 4. In this case we know that $s_i^*(v_i) = h_L$, so that $\bar{U}_i(v_i) = (v_i - h_L)$. Thus, $v_i < v_a(\lambda)$ implies

$$\begin{aligned} U_i^R(v_i, h_H) - U_i^N(v_i, h_H) &= \\ & (1 - \Pr[S_j]) \left(\int_{h_L}^1 \max\{v_i - s_j^*(y), 0\} d\Phi(y|\neg S_j) \right. \\ & \left. - 2 \int_{h_L}^{v_i} (v_i - y) d\Phi(y|\neg S_j) \right) \geq (1 - \Pr[S_j]) \\ & \left(\int_{h_L}^{v_a(\lambda)} (v_i - h_L) d\Phi(y|\neg S_j) - 2 \int_{h_L}^{v_i} (v_i - y) d\Phi(y|\neg S_j) \right) \equiv J(v_i) \end{aligned}$$

where the inequality follows from the fact that types of bidder j with a value in the interval $(h_L, v_a(\lambda))$ stop at h_L . Since $\int_{N_i \cap [h_L, v_a(\lambda)]} dG(y) > 0$, it follows that $\int_{h_L}^{v_a(\lambda)} d\Phi(y|\neg S_j) \equiv Z > 0$. Therefore we have

$$\frac{dJ(v_i)}{dv_i} = Z - 2\Phi(v_i|\neg S_j),$$

so that $\frac{dJ(v_i)}{dv_i} \Big|_{v_i=h_L} = Z > 0$. Therefore, there must be a non-empty interval $[h_L, v^*]$ such that reducing demand to one at h_L is optimal. ■

Proof of Proposition 4. Remember from Proposition that for any $\lambda > 0$, in any noncollusive equilibrium we have $v^*(\lambda) \geq v_a(\lambda)$. We will use this fact to show that as $\lambda \downarrow 0$, we must have $\lim_{\lambda \downarrow 0} v^*(\lambda) \geq v'$.

We first prove part 1 be contradiction. Suppose $v^*(\lambda) = v^* < v'$ for some λ , and consider types $v_i \leq v^*$. Define the constant

$$\xi_\lambda = \frac{1}{\lambda + (1 - \lambda) \int_{M_\lambda} dG(x)}$$

and observe that the density is

$$\phi_\lambda(y|S_j) = \xi_\lambda g(y) [\lambda + (1 - \lambda) 1_{M_\lambda}(y)].$$

Since $y \leq v^*$ implies $y \in M_\lambda$ we have

$$\phi_\lambda(y|S_j) = \xi_\lambda g(y) \quad y \leq v^*.$$

Thus the problem of finding the optimal stopping time for a type $v_i \leq v^*$ can be written as

$$\max_{s \in [h_L, v_i]} \int_{h_L}^s 2(v_i - y) \xi_\lambda g(y) dy + (v_i - s)[1 - \xi_\lambda G(s)].$$

This is a differentiable function. The first derivative is

$$\frac{\partial U}{\partial s} = (v_i - s) \xi_\lambda g(s) - [1 - \xi_\lambda G(s)]$$

and the second derivative is

$$\frac{\partial^2 U}{\partial s^2} = \xi_\lambda (v_i - s) g'(s) \geq 0.$$

Thus the first derivative is strictly negative at $s = v_i$, and nondecreasing for $s < v_i$. We conclude that h_L is the unique optimal stopping time for all types $v_i \leq v^*$.

Type v^* must be indifferent between reducing and not reducing demand to one at h_L , and we also know that $s^*(v^*) = h_L$ is the unique optimal stopping time. This implies

$$\int_{N_\lambda} \max\{v^* - s_j^*(y), 0\} dG(y) = 0,$$

so that $s_j^*(y) \geq v^*$ for almost all $y \in N_\lambda$. Since h_L is the unique optimal stopping time for type v^* there is a strictly positive constant κ such that

$$v^* - h_L = \int_{h_L}^{v^*} 2(v^* - y) \xi_\lambda dG(y) + \kappa$$

Now consider a type $v_i \in (\{v^*, v'\} \cap N_\lambda)$. There is a set of positive measure of such types, and v_i can be chosen arbitrarily close to v^* . Let $v_i = v^* + \delta$, and observe that δ can be taken to be arbitrarily small. The utility of v_i from

not reducing demand and stopping at $s \geq v^*$ is

$$U_i^N(s, v_i, h_H) \equiv \left[\lambda + (1 - \lambda) \int_{M_\lambda} dG(x) \right] U(s, v_i) + (1 - \lambda) \int_{N_\lambda} \\ \times 2 \max\{v_i - y, 0\} dG(y) \leq \left[\lambda + (1 - \lambda) \int_{M_\lambda} dG(x) \right] U(s, v_i) + (1 - \lambda) 2\delta$$

and

$$U(s, v_i) \equiv \int_{h_L}^s 2(v_i - y) d\Phi_\lambda(y|S_j) + (v_i - s) [1 - \Phi_\lambda(s|S_j)] \leq \int_{h_L}^{v^*} \\ 2(v^* - y) \xi_\lambda dG(y|S_j) + 3\delta = v^* - h_L - \kappa + 3\delta = v_i - h_L - (\kappa - 2\delta)$$

Thus

$$U_i^N(s, v_i, h_H) \leq \left[\lambda + (1 - \lambda) \int_{M_\lambda} dG(x) \right] [v_i - h_L - (\kappa - 2\delta)] \\ + (1 - \lambda)(v_i - v^*) [G(v_i) - G(v^*)]$$

Reducing demand to one yields at least

$$\left[\lambda + (1 - \lambda) \int_{M_\lambda} dG(x) \right] (v_i - h_L)$$

Thus, if

$$-\left[\lambda + (1 - \lambda) \int_{M_\lambda} dG(x) \right] (\kappa - 2\delta) + (1 - \lambda) \delta [G(v^* + \delta) - G(v^*)] < 0$$

for some δ we have found a contradiction. But this is clearly true for δ small enough, since $\lambda > 0$ and $\kappa > 0$.

We now prove part 2. By point 1, if a noncollusive equilibrium exists then it must have $v^*(\lambda) = 1$ for each $\lambda > 0$, i.e. $M_\lambda = [h_L, 1]$. We will now show that an equilibrium in fact exists.

The strategy of bidder i , type (v_i, h_i) is the following:

- If $v_i < h_L$ then keep two buttons pushed until the price reaches v_i , release both buttons at $p = v_i$.
- If $v_i \geq h_L$ then, no matter what the budget is, release one button at $p = h_L$ and release the second button at v_i .

The beliefs are given by Bayes' rule whenever possible. If a bidder releases just one button at any price $p \neq h_L$ then the belief is $\Phi(v|p) = 1_{[\{v \geq p\}]}(v)$.

Beliefs are clearly compatible with the strategy profile. We will only check that for a type (v_i, h_H) with $v_i > h_L$ it is optimal to reduce demand to one when the price reaches h_L ; the optimality of the strategy at other points is easy to check. On the equilibrium path, the utility obtained when the price

reaches h_L and demand is reduced to one is $(v_i - h_L)$. Since the opponent reduces demand to 1 for sure at h_L , the expected utility of not reducing demand to one is

$$\max_{s \in [h_L, v_i]} \int_{h_L}^s 2(v_i - y) dG(y) + (v_i - s)(1 - G(s))$$

Since f is nondecreasing on $[h_L, 1]$ it is easy to see that $s = h_L$ maximizes the expected utility. Therefore, the deviation is not profitable. ■

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