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SYMPOSIUM

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Simultaneous ascending auctions with complementarities and known budget constraints

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Abstract We study simultaneous ascending auctions of two identical objects when bidders are financially constrained and their valuations exhibit complementarities. We assume the budget constraints are known but the values for individual objects are private information, and characterize noncollusive equilibria. Equilibrium behavior is affected by the exposure problem. Bidders with higher budgets are more reluctant to bid, because opponents with lower budgets may end up pursuing a single object, thus preventing the realization of complementarities. Therefore poor bidders may win both objects when they do not have the highest valuation.

Keywords Multiple-object auctions · Budget constraints · Complementarities

JEL Classification Number D44 [H]

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1 Introduction

Since their introduction by the Federal Communication Commission in 1994, simultaneous ascending auctions have become a commonly used auction format for the sale of multiple objects. One reason for their popularity is that they allow bidders to adjust their bids between different objects as the auction progresses in light of the new information about the likelihood of obtaining different subsets of objects. This is particularly useful when significant subsets of the objects on sale tend to exhibit complementaries (see e.g. Szentes and Rosenthal 2003a) and the bidders have a limited amount of money for the auction, so that increasing the bid on one object decreases the amount of money available for bidding on other objects. The practical importance of budget constraints in the FCC auctions of spectrum licenses has been pointed out, among others, by Salant (1997).

The joint presence of complementarities and budget constraints makes the theoretical analysis of the equilibria of simultaneous ascending auctions particularly complex. The main difficulty is due to the so called ‘exposure problem’. This expression is used to indicate that by bidding ‘straightforwardly’—i.e. up to one’s valuation for all objects—the bidders expose themselves to the risk of having to buy only some elements in a set of complementary objects at prices that exceed their valuations. At least in some cases, the exposure problem induces bidders to stop bidding before prices arrive at their values.

In this paper we analyze the interplay between complementarities and budget constraints in simultaneous ascending auctions. In order to make the analysis tractable we restrict attention to the case of two identical objects and two bidders with known and different budget levels. The last assumption is a good approximation of reality in some instances, and it is a useful first step for the analysis of the case with privately known budgets. We also assume that each bidder is privately informed about both her ‘stand-alone value’ i.e. her willingness to pay a single unit, and her ‘complementarity premium’, i.e. the amount by which her value for the two unit bundle is larger than twice her stand-alone value.

We focus on ‘noncollusive’ equilibria, i.e. equilibria in which the bidders attempt to split the objects only when their budget constraints become binding. This is because when the bidders coordinate on ‘collusive’ equilibria, where each bidder buys a single object for a low price,[1] budget constraints are not binding and thus their presence has no impact on the outcome of the auction.

The equilibrium is characterized by various kinds of inefficiencies. In addition to the obvious direct effect, due to the fact that the low-budget bidder is sometimes unable to compete for the bundle, there are two indirect effects, which we label *monopsony effect* and *exposure problem*. The key to understanding these effects is that the presence of potentially binding budget constraints sometimes induces the ‘poor’ bidder to compete for only one object.

The monopsony effect comes from the fact that, once the poor bidder has reduced demand to one, the rich bidder has to decide whether to keep bidding for both objects or end the auction by reducing demand to one at the current price. Unless her complementarity premium is high, it is optimal for the rich bidder to inefficiently reduce demand to one before the price reaches her willingness to pay. This is a standard result in monopsony models. It is worth observing however that

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[1] See Brusco and Lopomo (2002) and Engelbrecht-Wiggans and Kahn (2005).

this effect would be present even without complementarities. In fact, the presence of complementarities tends to mitigate this problem.

The exposure problem is instead entirely due to the presence of complementarities. It comes from the fact that sometimes the high-budget bidder is better off dropping out of the auction altogether rather than taking the risk of buying only one object. It is worth emphasizing that it is the bidder with the highest budget who is affected by the problem. In general, a bidder is affected by the exposure problem when the *opponent* may decide to start bidding on one object only. Given the presence of complementarities, the only reason why a bidder might reduce demand to one in a noncollusive equilibrium is that his budget constraint becomes binding. Thus, it is always the ‘poor’ bidder who reduces demand to one, hence only the rich bidder can fear exposure. We will show that in equilibrium, due to the exposure problem, the rich bidder may leave the auction sooner than a poor bidder with the same value. Therefore, there are realizations of the bidders values for which both objects are assigned to the poor bidder, despite the fact that her valuation is lower.

To our knowledge, there is no paper that deals simultaneously with complementarities and budget constraints in ascending auctions. Instead, the literature has separately introduced complementarities or budget constraints in standard auction models.

The effect of complementarities in simultaneous auctions has been studied, among others, by Rosenthal and Wang (1996), Krishna and Rosenthal (1996), Szentes and Rosenthal (2003a,b), Englmaier et al. (2004), Fang and Parreiras (2002) and Chakraborty (2004). In all these papers the auction formats considered are not of the ascending type; rather, these papers focus on variants of first-price or second-price auctions. Our work is more related to Albano et al. (2006) and Zheng (2005). These papers consider ascending auctions in which some ‘global’ bidders have complementarities and want to pursue both objects while other ‘local’ bidders pursue a single object. The identities of global and local bidders are common knowledge, although the stand-alone values and the complementarities are private information. Neither Albano et al. (2006) nor Zheng (2005) consider the presence of budget constraints, as we do. Another important difference is that we consider bidders with *ex ante* identical values; i.e. both stand alone values and complementarities are drawn from the same distributions, and there is no distinction between ‘local’ and ‘global’ bidders.

The effects of budget constraints for various auction formats have been first studied in a seminal paper by Che and Gale (1998). Under the assumption of complete information, Benoît and Krishna (2001) have studied the effect of budget constraints on sequential auctions, and Cho et al. (2002) have analyzed a single-object first-price auction model in which liquidity constrained bidders pool their financial resources by endogenously forming coalitions before bidding. Finally, in Brusco and Lopomo (2002) we have studied simultaneous ascending auctions without budget constraints with heterogenous objects and either zero or large complementarities, and in Brusco and Lopomo (2004) we have analyzed simultaneous ascending auctions with privately known budget constraints and homogeneous objects, but without complementarities.

The rest of the paper is organized as follows. In Sect. 2 we specify the rules of the auction and the assumptions on the bidders’ preferences. In Sect. 3 we characterize the noncollusive equilibrium. Section 4 contains concluding remarks. All the proofs are in the Appendix.

2 The model

There are two identical units of a good and two risk neutral bidders. Bidder $i \in \{1, 2\}$ is willing to pay v_i for a single unit, and $2(v_i + k_i)$, $k_i \geq 0$, for both units. We will refer to the variables v_i and k_i as bidder i's 'stand-alone value' and 'complementarity premium' respectively (note that k_i is the *per-unit* premium). Bidder i's surplus when she obtains $n \in \{1, 2\}$ objects and pays a total amount of m is

$$U_i(n, m | v_i, k_i) = \begin{cases} v_i - m, & \text{if } n = 1, \\ \\ 2(v_i + k_i) - m, & \text{if } n = 2. \end{cases}$$

The four variables (v_1, v_2, k_1, k_2) are distributed independently, with support $[0, 1]^2 \times \left[\underline{k}, \overline{k}\right]^2$, where $\underline{k} \geq 0$. The stand-alone values v_1, v_2 are identically distributed, with c.d.f. F and differentiable density f, and the complementarity premia k_1, k_2 are identically distributed, with c.d.f. G and differentiable density g. The realization of the pair (v_i, k_i) is privately known to bidder i at the beginning of the auction.

Bidder i's total payment cannot exceed her 'budget' w_i. Both budget levels, w_1 and w_2, are common knowledge. We assume that $w_1 \neq w_2$, and without additional loss of generality we let bidders 1 and 2 be the 'poor' and 'rich' bidder respectively, i.e. we set $w_1 < w_2$.

To simplify the analysis without altering the substance of our results we also assume that $1 < w_1$, so that even the poor bidder can always bid up to her stand-alone value, i.e. $v_1 < w_1$. However, even bidder 2 may be unable to to bid up to her valuation for the two-unit bundle; that is we allow for the possibility that $w_2 < 2(v_2 + k_2)$. It is convenient to define $h_i \equiv \frac{w_i}{2}$; this is the highest unit price at which bidder i can bid on both objects.

The objects are sold using a 'simultaneous ascending clock auction' (SACA) working as follows. Each bidder is given two buttons. There is a single price which starts at zero and increases at constant speed, until at least one bidder lifts at least one button. The general idea is that lifting l buttons at price p means demanding l fewer units at any price *strictly* higher than p, until additional buttons are lifted. Thus by lifting one button at price p, and the second button at $p' > p$, bidder i communicates that she is willing to pay up to $2p$ for two units, and up to p' for one unit.

Demand reduction is irreversible: once released, a button cannot be pushed again. This is the simplest version of the *activity rules* that are often used in simultaneous ascending auctions.[2]

The continuous time format of the auction requires care in the specification of some details, in order to make sure that the resulting game form is well defined. As

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[2] In most of the FCC auctions used for the sale of spectrum licenses, some variation of the following basic rule were put in effect: "a bidder that places eligible bids for n units at round t cannot place bids for more than n units at any subsequent round $t' > t$". (Milgrom 2004).

Table 1 Objects' allocation

2	0,2	0,2	$\frac{1}{2}(2, 0)$ $+\frac{1}{2}(0, 2)$
1	δ-pause	1,1	2,0
0	/	δ-pause	2,0
	0	1	2

usual, the technical problem is that sometimes a player wants to react to an action by her opponent as soon as possible, and this may create an 'open set problem'. In our case this happens, for example, when the price is increasing and bidder 2 wants to end the auction as soon as possible after bidder 1 lifts one button; that is, if bidder 1 lifts one button at time τ, then bidder 2 wants to lift one of her buttons at the lowest time t such that $t > \tau$. This problem has no solution because the constraint set is open.

To get around this issue, we adopt a slightly modified version of the auction format proposed by Zheng (2005). We specify that when one [3] button is lifted for the first time, say by bidder 1, the price stops for an interval of time δ. During this period of time bidder 2 is allowed to reduce her demand at the same price. In addition, we want to allow bidder 1 to react to bidder 2's reaction before the price can resume its upward movement.

The formal specification of the rules is as follows. Suppose that bidder i is the first to lift exactly one button at time t, when the price is p_t. Then the price stops raising, and bidder $j \neq i$ is asked whether she wants to react by lifting one or two buttons. If j lifts any button, the auction ends; otherwise bidder i is asked whether she wants to lift her second button. If she does, the auction ends; and if she does not, the price resumes its upward movement starting from p_t, with bidder i pushing one button and bidder j pushing two buttons. Thus the price may start moving again only after both bidders have had a chance to react to the status quo and have chosen to do nothing.

The allocation of the objects and the sale price are determined as follows. Let t denote the first time at which bidder i reduces demand, i.e. releases one or two buttons. If bidder i lifts two buttons, then her opponent $j \neq i$ buys both objects at unit price p_t, unless j also releases two buttons at t, in which case the tie is broken by assigning the two-item bundle to each bidder with probability $\frac{1}{2}$. These cases correspond to the cells in the top row and right column of Table 1 below, which illustrates the objects' allocation for any combinations of buttons lifted by the two bidders at time t. More precisely, the first number in each cell denotes the number of objects that are assigned to the column bidder, and the second is the number of objects assigned to the row bidder. The numbers 0, 1 and 2 along the sides denote the number of buttons lifted by each bidder.

If both bidders release exactly one button at time t, then each bidder buys one object at price p_t. This is the central cell in Table 1.

Finally, if bidder i releases one button and j does nothing at t, the price stops for an interval of time δ, during which j is given the opportunity to react: if j releases one button, the auction ends with each bidder buying one unit at p_t; if j releases

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[3] If more than one button is released, the auction ends.

two buttons, then i buys both objects at p_t each;[4] and if j does nothing, then i is given a chance to release her second button in which case j gets both objects at p_t each, otherwise the price resumes its upward movement, starting from $p_{t+\delta} = p_t$.

In this last case, the auction ends as soon as any bidder releases another button: if bidder j releases one or two buttons then each bidder buys one object at the current price; if instead bidder i releases her second button, bidder j wins both objects.

A few remarks on the auction rules are in order. First note that, differently from Zheng (2005), we have a single price and the buttons are not object–specific. In contrast, Zheng has separate auctions, each with its separate price. This is an important difference, since in Zheng the auction format also allows for jump bidding. The reason why this is important is that Zheng is able to prove that the combination of different prices and jump bidding allows the bidders to signal their values and eliminate some of the inefficiency that may appear when goods exhibiting complementarities are sold in separate markets; this possibility is ruled out in our auction format with a single price.

In contrast, our auction format is equivalent to one in which any bidder can resume bidding on any object, even if she has not done so continuously since the beginning of the auction, as long as her bidding activity (the number of objects on which she is bidding) does not increase. This format, with the activity rule, is intermediate between the case where exit on each object is irrevocable and the case with unrestricted re-entry.

Second, the tie-breaking rules are designed to maximize the probability of realizing the complementarities, i.e. of assigning both objects to the same bidder. This is why the two-unit bundle is allocated to one (randomly chosen) bidder when both bidders reduce their demand to zero simultaneously. Also, when one bidder reacts to the demand reduction of another bidder by lifting two buttons, both objects are allocated to the bidder who reduced demand to one. By maximizing the probability of assigning both objects to a single bidder we minimize the inefficiencies caused by the presence of potentially binding budget constraints. If the tie-breaking rules were changed the equilibria described in the next section would remain qualitatively the same, but would entail additional distortions from the efficient outcome.

In our analysis we rule out (weakly) dominated strategies, hence we focus on equilibria where no bidder reduces demand to zero if the price is less than her stand-alone value. Moreover, since our main goal is to study the impact that the simultaneous presence of budget constraints and complementarities has on the level of efficiency that can be achieved in the SACA, we focus on ‘noncollusive’ equilibria, i.e. equilibria in which the bidders bid ‘straightforwardly’ as much as possible. Equilibria with a collusive flavor, in which the bidders manage to coordinate on buying one object each for a low price, can be constructed as follows.[5] Suppose that bidder 1 reduces demand to one when the price is 0, implicitly inviting the opponent to split the objects. If this ‘offer’ is declined (i.e. if bidder 2 does not release any of her buttons), the only optimal continuation strategy for bidder 1 is to bid on a single unit up to her value v_1, in which case bidder 2’s optimal continuation

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[4] This is because, as mentioned at the beginning of the section, lifting one button at p_t indicates willingness to buy both units for p_t each, and at most one unit for any price above p_t.

[5] In Brusco and Lopomo (2002) we have studied collusive behavior in simultaneous ascending auctions

strategy is to release one object at an optimally chosen ‘stopping time’. For a wide class of distributions of v_i and k_i, splitting the objects at $p = 0$ is an equilibrium. Moreover, there may exist similar equilibria inducing the splitting of the objects at other prices below h_1.

We ignore collusive equilibria here because their existence does not hinge on the presence of budget constraints. Our goal is to identify the distortions from the first best that arise in the SACA when both budget constraints and complementarities are present. Thus we only consider equilibria in ‘noncollusive’ strategies, according to which bidder i bids on both objects as long as $p \leq \min\{h_1, v_i + k_i\}$, i.e. unless either the implied payment is more than her value for the bundle—$2p > 2(v_i + k_i)$—or the price is above h_1, in which case bidder 1 cannot bid on both objects, because her budget constraint is now binding.

We begin by establishing an easy benchmark: without budget constraints, the SACA has a ‘bundling’ equilibrium in which both objects always go to the bidder with the highest total value, thus implementing the efficient allocation.[6]

Proposition 1 *If* $2\left(1+\overline{k}\right) < w_i$ *for each* $i \in \{1, 2\}$, *there exists a perfect Bayesian equilibrium in which bidder i wins both objects whenever* $v_i + k_i > v_{-i} + k_{-i}$.

In a ‘bundling’ equilibrium each bidder bids straightforwardly, demanding both objects when $p \leq v_i + k_i$ and zero otherwise. This is always feasible under the assumption that the type with the highest value for the bundle $2\left(1+\overline{k}\right)$ can do so. Thus on the equilibrium path the bidders only compete for the two-unit bundle. To guarantee that demanding one unit is never a profitable deviation, we select beliefs that assign high probability to low values for any bidder who lifts only one button. Thus if bidder i releases only one button (an out of equilibrium action) her opponent will not accept to split the objects and will instead keep bidding on both objects up to her value, because she expects to pay a low price. This makes the deviation unprofitable.

Proposition 1 shows that there exists an equilibrium in which the exposure problem does not arise when both bidders are not budegt constrained. Therefore the sole presence of complementarities is not sufficient in our setting to generate any distortion from efficiency. [7] The distortions from the efficient outcome that we are going to find necessarily arise only when both complementarities and potentially binding budget constraints are present.

These distortions are caused by three effects. First, there is an obvious direct effect due to the fact that in some cases the bidder with the higher value for the two-unit bundle cannot afford to pay for both objects, hence the objects end up being split. The bidder who suffers more from this effect is the one with the lower budget.

A second, more interesting effect, is caused by the exposure problem that is created for the high budget bidder, i.e. bidder 2. As we will see, in any noncollusive equilibrium there is a nonnull set of types of bidder 2 who lift *both* buttons before

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[6] Chakraborty (2004) discusses conditions under which various simultaneous (but non-ascending) auctions may have ‘bundling equilibria’, i.e. equilibria in which the strategies are such that a bidder either gets the bundle or nothing, so that the exposure problem does not arise.

[7] The exposure problem would reappear however if the objects were heterogeneous, (e.g. if a bidder were only interested in one of the objects), even without budget constraints.

the total payment implied by the current price arrives at her value for the bundle. This happens when the price is between v_2 and $v_2 + k_2$, and is due to the bidder's fear of ending up having to buy a single object and thus earning negative surplus. The loss of social surplus in this case is not due to the fact that the objects are split, but rather to the fact that the bundle may be assigned to the bidder with the lower value. Interestingly enough, the bidder who is hurt by this second effect is the one with the *higher* budget, because it is the 'poor' bidder who is the first to reduce demand to one unit, thus creating the exposure problem for the 'rich' opponent.

Finally there is a 'monopsony effect' that arises after bidder 1 lifts one button at h_1. We will show that this also happens in any noncollusive equilibrium. In this case bidder 2 faces a classic monopsony trade-off between quantity (one versus two units) and price, and thus will generally lift one button before the total payment implied by the price reaches her value for the two-unit bundle. Therefore the objects may end up being split even if bidder 2 is not budget constrained and has a higher value for the bundle. It is worth pointing out that this effect will in general be moderated by the presence of complementarities, which make pursuing both objects more attractive.

3 The noncollusive equilibrium

In any equilibrium in undominated strategies, bidder i can win both objects only if her opponent's stand-alone value does not exceed half or her budget, i.e. $v_j \leq h_i$. This is because bidder j reduces demand to zero only after the price becomes larger than v_j; and if $h_i < v_j$, bidder i cannot afford to pay $p \geq v_j > h_i$ for each object. When this happens, and $v_j + k_j < v_i + k_i$, the allocation will be inefficient. This is the direct effect of budget constraints, which generates an upper bound on the level of efficiency that can be attained with a simultaneous ascending auction.

In this section we show that this upper bound cannot be attained by any equilibrium of the SACA. This is due to both the exposure problem and the monopsony effect, which induce bidder 2 to reduce her demand, to zero and one respectively, before the total payment implied by the price arrives at her value. It is worth pointing out that, while the exposure problem only arises in the presence of complementarities, the monopsony effect is most severe when bidder 2's complementarity premium k_2 is zero, as the incentive to buy the second unit increases with k_2. Thus the overall impact that the presence of complementarities has on the level of efficiency with budget constrained bidders is ambiguous: complementarities create the exposure problem, but also mitigate the monopsony effect.

To see how these two additional effects appear in any noncollusive equilibrium, first observe that straightforward bidding pushes the price to h_1 when both bidders have sufficiently high values, i.e. $v_i + k_i \geq h_1$ for each i. Once the price arrives at h_1, bidder 1 must reduce her demand, as her budget constraint becomes binding. The next lemma establishes, that the only equilibrium continuation strategy is to reduce demand to one and then, if bidder 1 keeps demanding both objects, lift the second button before the price resumes its upward movement or bid on one unit up to v_1, depending on whether her stand-alone value is below or above h_1.

Lemma 1 *In any noncollusive equilibrium, all types of bidder 1 with* $h_1 < v_1 + k_1$ *reduce their demand to one at* h_1*. If the opponent does not react, then they leave the auction when* $v_1 < h_1$*, and bid on one unit up to* v_1*, if* $v_1 > h_1$*.*

In light of Lemma 1 it is easy to characterize the set of all equilibrium continuation strategies for bidder 2. Recall that, in equilibrium, once the price has arrived at h_1, it is common knowledge that $v_i + k_i \geq h_1$ for each $i = 1, 2$. After bidder 1 stops the price by lifting one button, bidder 2 can reduce demand to zero before the price starts moving again, thus earning zero surplus, or do nothing until the price moves again and arrives at any level $p \in [h_1, h_2]$, and then lift one button. The latter strategy yields both objects at unit price v_1 if $v_1 < p$, and one object at price p if $v_1 > p$, hence an expected surplus of

$$V(p|v_2, k_2) \equiv \int_{h_1}^{p} 2(v_2 + k_2 - v_1)\,\mathrm{d}G(v_1) + [1 - G(p)](v_2 - p), \qquad (1)$$

where $G(v_1) \equiv \frac{F(v_1) - F(h_1)}{1 - F(h_1)}$, since conditional on v_1 competing on one object it must be the case that $v_1 > h_1$. It is easy to see that, as $v_2 + k_2$ approaches h_1, the first term on the right hand side of (1) goes to zero, while the second term becomes negative. Thus, for $v_2 + k_2$ sufficiently close to h_1 we have $V(p|v_2, k_2) < 0$ for all $p \in [h_1, h_2]$. Therefore there exists a set of types of positive measure with $v_2 + k_2 > h_1$ for whom it is optimal to lift both buttons before the price starts raising again. These are the types whose behavior is affected by the exposure problem. In this case bidder 1 wins both objects, and the resulting allocation is inefficient whenever $v_1 + k_1 < v_2 + k_2$.

For all other types of bidder 2, any equilibrium continuation strategy is characterized by an optimal 'stopping time'

$$p^*(v_2, k_2) \quad \in \arg\max_{p \in [h_1, h_2]} V(p|v_2, k_2), \qquad (2)$$

which is often strictly below bidder 2's willingness to pay $v_2 + k_2$. To see this, note that the first derivative

$$\left.\frac{\partial V(p|v_2, k_2)}{\partial p}\right|_{p = v_2 + k_2} = G'(p)\,k_2 - [1 - G(v_2 + k_2)]$$

is negative for k_2 sufficiently small.

3.1 Partition strategies

We now proceed to complete the characterization of the noncollusive equilibrium. To simplify the exposition, we assume $h_1 > \underline{k}$; the case $h_1 \leq \underline{k}$ can be easily accommodated. We define, and illustrate in Figs. 1 and 2, a pair of *partition strategies* as follows.

Player 1 (low budget player)

- Types (v_1, k_1) such that $v_1 < h_1 - k_1$ (set A in Fig. 1) bid on both objects up to $v_1 + k_1$ and then reduce their demand to zero.

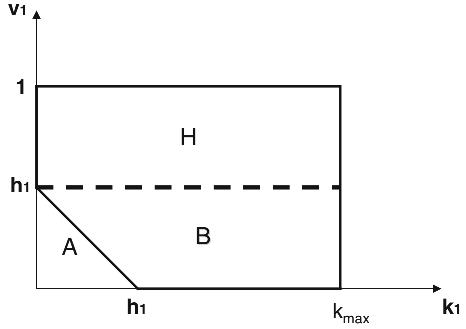

Fig. 1 Low-budget bidder's partition

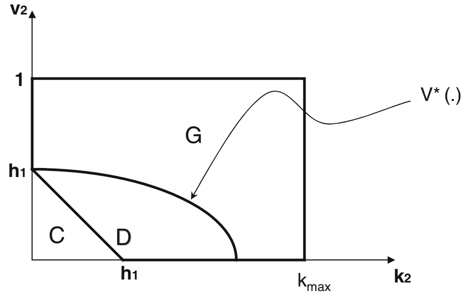

Fig. 2 High-budget bidder's partition

- Types (v_1, k_1) such that $h_1 - k_1 < v_1 < h_1$ (set B in Fig. 1) bid on both objects up to h_1 and then reduce demand to one; if bidder 2 does not react, they reduce demand to 0.
- Types (v_1, k_1) such that $v_1 > h_1$ (set H in Fig. 1) bid on both objects up to h_1, then bid on one object up to v_1.

Player 2 (high budget player)

- Types (v_2, k_2) such that $v_2 < h_1 - k_2$ (set C in Fig. 2) bid on both objects up to $v_2 + k_2$ and then reduce their demand to zero.
- Types (v_2, k_2) such that $h_1 - k_2 < v_2 < v^*(k_2)$, where the function $v^* : [\underline{k}, \overline{k}] \to [0, h_1]$ is determined by Eq. (4) below, (set D in Fig. 2) bid on both objects up to h_1 and then, when bidder 1 reduces demand to 1, react by reducing demand to 0.
- Types (v_2, k_2) such that $v_2 > v^*(k_2)$ (set G in Fig. 2) bid on both objects up to h_1 and then, when bidder 1 reduces demand to 1, do not react. If the opponent remains in the auction, they keep their demand at 2 until an optimal 'stopping time' $p^*(v_2, k_2)$ (determined by (3) below) and then reduce demand to one, thus ending the auction.

The strategy of the low–budget bidder can be loosely described as follows. First, all types in set A, with a sufficiently low value for the bundle ($v_1 + k_1 \leq h_1$) bid straightforwardly, demanding two objects until the price reaches $v_1 + k_1$ and then dropping both objects. Second, all types in set B, with a sufficiently high value for the bundle ($v_1 + k_1 > h_1$) but a relatively low stand-alone value ($v_1 \leq h_1$) try to buy both objects until the price reaches h_1, and at that point lift one button, hoping that the opponent ends the auction. If instead the opponent does not react, thus keeping the demand at 2, they leave the auction, since their stand-alone value is inferior to the price. Finally, all types in set H, with both a high value for the bundle ($v_1 + k_1 > h_1$) and a high stand-alone value ($v_1 > h_1$) behave much in the same way, except that they keep trying to buy a single object until the price reaches v_1. Note that these are exactly the types that bidder 2 fears the most, since they are the ones who can get bidder 2 exposed.

Consider now the strategy of the high-budget bidder. It is fairly obvious that all types in set C, with a low value for the bundle ($v_2 + k_2 \leq h_1$) should bid straightforwardly, as there is no risk of exposure until the price reaches h_1.

The optimal strategy for the types in $D \cup G$, with a high value for the bundle ($v_2 + k_2 > h_1$) is more complicated, since when the price reaches h_1 the exposure problem appears. The partition of this set in D and G strategy is determined as follows. The behavior of all types in D, with a relatively low stand-alone value, i.e. $v_2 \in \left(h_1 - k_2, v_2^* (k_2)\right]$, is affected by the exposure problem: they demand two objects until the price reaches h_1, and then, when bidder 1 reduces demand to 1, they leave the auction. Note that this can happen even if bidder 1 has a lower value.

All types in G, with a high stand-alone value, i.e. $v_2 > v_2^* (k_2)$, are less affected by the exposure problem. They keep trying to buy both objects even when the opponent has reduced demand to one, and do so up to an optimally chosen stopping price $p^* (v_2, k_2)$. At that point they reduce demand to one, thus splitting the objects and ending the auction.

To define the details of bidder 2's strategy, consider her decision problem when the price has arrived at h_1 and the opponent has reduced demand to one. Reducing demand to zero at this point yields zero surplus because, according to the tie-breaking rules, the bundle is assigned to bidder 1.

The payoff resulting from keeping both buttons pushed instead depends on bidder 1's reaction: if she leaves the auction (this happens when $v_1 \leq h_1$) bidder 2 earns $2 (v_2 + k_2 - h_1)$. If instead bidder 1 remains in the auction (when $v_1 > h_1$), bidder 2 is forced to buy at least one object, and her optimal 'stopping time' is obtained solving the problem[8]

$$\max_{p \in [h_1, h_2]} \int_{h_1}^{p} 2 (v_2 + k_2 - v_1)\, \mathrm{d}G(v_1) + [1 - G(p)] (v_2 - p), \tag{3}$$

Let $V (v_2, k_2)$ denote the value of the objective function at the optimal point.[9]

Now define

$$\xi \equiv Pr (v_1 \leq h_1 | v_1 + k_1 \geq h_1).$$

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[8] Note that, once the bidder 1 has decided to remain in the auction, only the distribution of v_1 matters for bidder 2, and this is given by $G (v_1) = F (v_1 | v_1 \geq h_1)$.

[9] It cannot be optimal for bidder 2 to reduce demand to one after the opponent has reduced demand to one. This is because bidder 2 can do better by not reacting, waiting to see if the opponent leaves the auction, and if the auction continues, set a stopping price $p = h_1$, i.e. end the auction immediately once it is clear that the opponent will keep bidding.

This is simply the ratio between the probability of set B and the probability of set $B \cup H$ in Fig. 1. The expected utility for bidder 2 of continuing to push two buttons after the opponent has reduced demand to one is

$$H(v_2, k_2) \equiv \xi\,[2(v_2 + k_2 - h_1)] + (1 - \xi)\, V(v_2, k_2).$$

For each fixed value k_2, the function $H(\cdot, k_2)$ is continuous and strictly increasing function in v_2, strictly negative at $v_2 = h_1 - k_2$ and strictly positive at $v_2 = h_1$. Thus, the value $v^*(k_2)$ is obtained as the unique solution to the equation

$$H(y, k_2) = 0, \tag{4}$$

and belongs to the interval $[h_1 - k_2, h_1]$. Types with a stand alone value $v_2 < v^*(k_2)$ are better off leaving the auction immediately. This is because lifting both buttons guarantees a payoff of zero, while not doing so creates the possibility of buying a single object and thus earning a negative payoff which is too large, given the odds entailed by ξ. On the other hand, types with $v_2 > v_2^*(k_2)$ obtain a strictly positive expected utility if they continue to compete in the auction. Thus it is optimal for them to continue demanding both objects until the price reaches $p^*(v_2, k_2)$, the optimal stopping price obtained solving (3), and then they reduce demand to one thus ending the auction.

The function $v^*(k_2)$ can be characterized as follows. First, when $k_2 = 0$ we have $v^*(0) = h_1$. Second, the function is decreasing, since $\frac{\partial H}{\partial k_2} > 0$. Third, there is a value $\widehat{k_2}$ such that $H\left(0, \widehat{k_2}\right) = 0$. If $\widehat{k_2} < \overline{k}$ then all types (v_2, k_2) with $k_2 \geq \widehat{k_2}$ will not react when bidder 1 lifts one button (this is the case pictured in Fig. 2).

The partition strategies can be used to construct a perfect Bayesian equilibrium, as recorded in the next proposition.

Proposition 2 *There exists a perfect Bayesian equilibrium in which the bidders adopt the partition strategies. This is the only non-collusive equilibrium.*

The discussion before the proposition implies that, by construction, partition strategies are optimal along the equilibrium path. The proof of the proposition completes the description of the equilibrium by finding beliefs and strategies out of the equilibrium path making sure that no deviation from the equilibrium path can be optimal.

The *equilibrium outcome* can be described as follows.

1. When $\min\{v_1 + k_1, v_2 + k_2\} \leq h_1$ (bidder 1's type is in A and/or bidder 2's type is in C), the outcome is the same as in the bundling equilibrium described in Proposition 1. In this case, the value for the bundle of at least one bidder is so low that budget constraints are never binding. The auction ends with bidder i buying both objects and paying $v_{-i} + k_{-i}$, whenever $v_i + k_i > v_{-i} + k_{-i}$. The outcome in this case is efficient.
2. When $h_1 < v_1 + k_1 < 1$ and $h_1 - k_2 < v_2 < v^*(k_2)$ (bidder 1's type is in B or H and bidder 2's type is in D), bidder 1 reduces demand to 1 when the price reaches h_1 and bidder 2 reacts by reducing demand to zero. Thus bidder 1 buys both objects at a price h_1. This may happen *despite the fact that* $v_1 + k_1 < v_2 + k_2$. In this case, the bidder with the lower budget *and* the lower value for the bundle is able to buy the bundle. Note also that in this region

the objects are never split. The only source of inefficiency is that the bundle is sometimes assigned to the poor bidder even if $v_1 + k_1 < v_2 + k_2$. In this case, the inefficiency is due entirely to the exposure problem.

3. When $h_1 - k_1 < v_1 < h_1$ and $v_2^* (k_2) < v_2 < 1$ (bidder 1's type is in B and bidder 2's type is in G) , bidder 1 reduces demand to 1 when the price reaches h_1. Bidder 2 does not react, and bidder 1 releases the second button. Thus bidder 2 buys both objects at h_1.
4. When $h_1 < v_1 < 1$ and $v_2^* (k_2) < v_2 < 1$ (bidder 1's type is in H and bidder 2's type is in G), bidder 1 reduces demand to 1 when the price reaches h_1, bidder 2 does not react, and bidder 1 bids up to v_1 on a single object. The auction continues until the price reaches the lowest of v_1 (in which case bidder 2 wins both objects) or the optimal stopping time $p^* (v_2, k_2)$ (in which case the objects are split).

As previously observed, the outcome of the auction is inefficient because of both the exposure problem and the monopsony effect. How important are these effects? Note first that, since $v^* (k_2)$ is decreasing, the exposure problem becomes less severe as the value of the complementarities increases. This is intuitive. When k_2 becomes larger the expected cost of getting a single object remains constant, but the expected benefit of getting the bundle goes up. Thus, more types will be willing to take the risk of pursuing the bundle.

The extent of the exposure problem can be roughly measured by the area of set D, in Fig. 2. In general, the shape of the function $v^* (k_2)$ depends on the parameters of the model, but it is interesting to observe what happens when h_2 changes. When h_2 increases, the value $V (v_2, k_2)$ (weakly) increases for each pair (v_2, k_2), since the constraint set expands. In fact, the only role played by h_2 is to constrain the choice of p; there may be situations in which the rich bidder would like to keep bidding on both objects but is prevented from doing so by its budget. Therefore the expected value of letting the auction continue increases, and the value $v^* (k_2)$ (weakly) decreases. For each value of k_2 , the highest value of $V (v_2, k_2)$ is obtained by setting $h_2 \geq 1$ in (3). The resulting expression for $V (v_2, k_2)$ can then be plugged into Eq. (4); the solution to the equation represents the lowest value of $v^* (k_2)$.

The highest value of $v^* (k_2)$ is instead obtained when h_2 converges to h_1 from above. In this case it becomes pointless for bidder 2 to try to get both objects, since the tight budget will force bidder 2 to reduce demand to one very quickly after the price has passed h_1. In that case $V (v_2, k_2)$ converges to $[1 - G (h_1)] (v_2 - h_1) = v_2 - h_1$, since $G (h_1)$ goes to 0. Therefore, $v^* (k_2)$ converges to the straight line

$$\overline{v}_2 (k_2) = h_1 - \frac{2\xi}{1 + \xi} k_2.$$

Note also that $\frac{2\xi}{1+\xi} < 1$, hence D is non–empty even as h_2 tends to h_1 from above.

A last point that deserves discussion is the effect of a change in the budget of the poor bidder. Specifically, is bidder 1 worse off or better off when h_1 is increased? A change in h_1 changes the various areas in the partition equilibrium. The changes in Fig. 1 are straightforward: the area of set A expands and the area of set H shrinks. For bidder 2 things are more complicated. While region C expands, the boundaries of D and G are determined by the distribution $F (v_1 | \, v_1 \geq h_1)$ and by the conditional probability $\Pr (v_1 \leq h_1 | v_1 + k_1 \geq h_1)$. Without additional information on the distribution it's impossible to tell how these areas change.

However it is not difficult to see that at least some types of bidder 1 can be hurt by an increase in h_1. Consider two budget levels $h_1 < h_1'$, and consider a type (v_1, k_1) such that $h_1 + \varepsilon = v_1 + k_1 < h_1'$ and $v_1 < h_1$, with ε small. When the budget is h_1' the utility of this type is simply

$$\int_0^{v_1+k_1} 2(v_1 + k_1 - s)\, \mathrm{d}H(s) = \int_0^{h_1+\varepsilon} 2(h_1 + \varepsilon - s)\, \mathrm{d}H(s),$$

where $H(\cdot)$ is the distribution of $v_2 + k_2$. When the budget is h_1 then the utility is

$$\int_0^{h_1} 2(v_1 + k_1 - s)\, \mathrm{d}H(s) + 2(v_1 + k_1 - h_1)\, Pr\left[(v_2, k_2) \in D_{h_1}\right]$$
$$= \int_0^{h_1} 2(h_1 + \varepsilon - s)\, \mathrm{d}H(s) + 2\varepsilon Pr\left[(v_2, k_2) \in D_{h_1}\right],$$

where D_{h_1} is the set D in the partition equilibrium when the budget of the poor bidder is $w_1 = 2h_1$. Therefore, this type of the poor bidder is better off with a smaller budget if

$$Pr\left[(v_2, k_2) \in D_{h_1}\right] > \frac{1}{\varepsilon} \int_{h_1}^{h_1+\varepsilon} (h_1 + \varepsilon - s)\, dH(s).$$

The left hand side is strictly positive and it does not depend on ε, while the right hand side converges to zero as ε goes to zero (it is just the derivative of $\int_{h_1}^{x} (x - s)\, \mathrm{d}H(s)$ with respect to x computed at $x = h_1$). Thus, types (v_1, k_1) such that $v_1 < h_1$ and $v_1 + k_1$ is sufficiently close to h_1 will be made worse off by a change in the budget. If sufficient *ex ante* probability is put on types who are made worse off, then a poor bidder may end up with a lower expected utility when the budget is increased. Suppose now that a bidder has to decide whether to make an effort to increase the budget against a richer opponent before learning the type (v_1, k_1). Our analysis suggests that, in some cases, the poor bidder will prefer a lower budget. Of course there are also situations in which a larger budget will help the poor bidder. A trivial case is one in which $w_1 = 0$, so that increasing the budget obviously increases the expected utility.

As a last remark, we observe that our analysis has been limited to a specific auction format. In particular, we have not investigated the optimal design of mechanisms when bidders are budget constrained. In general, it is difficult to implement efficient allocations when bidders have multi-dimensional types. We refer the reader to Jehiel et al. (1999) and Maskin (2000).

3.2 Discussion and extensions

In this section we discuss the robustness of our findings to: (i) minor changes in the auction rules, and (ii) the introduction of additional bidders and/or objects.

3.2.1 Auction rules

Our analysis is tied to a specific auction format. In particular, the crucial assumptions are that the two objects are sold at the same price and the price increases as long as there is excess demand. The specific details of the equilibrium also depend on the particular activity rule considered; however, the main features of the equilibrium are robust to changes in the activity rules. In fact, as pointed out in the introduction, the activity rule has been chosen in order to maximize the probability of selling the bundle to a single bidder, and thus to minimize the exposure problem.

Suppose, for example, that we change the rule as follows: After one bidder has reduced demand to 1, if the other bidder reacts by reducing demand to zero then the objects are split (instead of both going to the bidder who is still demanding one object). Finding a perfect Bayesian equilibrium for this auction format is not immediate. However it is easy to see that the exposure problem must still have an impact on the equilibrium, i.e. some bidder must leave the auction with zero objects when the price is below his value for the bundle.

Suppose that the poor bidder bids straightforwardly up to h_1, as it should happen in a noncollusive equilibrium. When the price reaches h_1 all types with $v_1 < h_1$ reduce demand to zero, while types with $v_1 > h_1$ reduce demand to one. Once the poor bidder has reduced demand to one, the problem for the rich bidder is the same as before, so an optimal stopping time is determined. Now consider the types of bidder 2 that are in set D. These types have a negative value for $V(v_2, k_2)$ from continuation. Therefore they are now better off by leaving the auction *right before*[10] the price reaches h_1. In fact, suppose that the price is $h_1 - \delta$, with δ small. Leaving the auction at this point gives a utility of zero, while staying in the auction until h_1 gives approximately a utility equal to

$$H_\delta(v_2, k_2) = \xi_\delta 2(v_2 + k_2 - h_1) + (1 - \xi_\delta) V(v_2, k_2),$$

where $\xi_\delta \equiv \Pr(v_1 \leq h_1 | v_1 + k_1 \geq h_1 - \delta)$. Thus, as δ goes to zero the set of types who decide to leave the auction before the price reaches h_1 converges to D. In order to assess the full extent of the exposure problem we would have to compute the equilibrium under the new activity rule. It is clear however that the presence of the exposure problem is not linked to the particular activity rule chosen.

Another possible change in the auction rules is to allow for jump bidding, as in some versions of the Zheng (2005) model. In his model, there are three bidders and two different objects, call them A and B, which are also priced separately. Among the bidders, one (the global bidder) attaches value to the bundle only, and zero value to each single object; while the remaining two bidders are ‘local’, i.e. they assign value only to A and B, respectively, and have no use for the other object. Jump bidding in this model eliminates the exposure problem.[11] The idea is the following. In this framework the exposure problem can only arise because the global bidder may be reluctant to bid on, say, object B when one local bidder stops

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[10] Of course ‘right before’ is not well defined with this auction format. For a rigorous analysis of the equilibrium in this case, some other changes should be introduced in order to avoid the ‘open set’ problems discussed in Sect. 1.

[11] Jump bidding does not eliminate all inefficiencies. Overconcentration, i.e. having the global bidder buying both objects when the sum of the valuations of the local bidders is higher, is still possible.

bidding on object A. The problem is that the local bidder on B can push the price to a point at which the value of the bundle for the global bidder is inferior to the sum of the prices. Jump bidding eliminates the problem by giving to the bidders an opportunity to signal their value. Thus, if one local bidder stops bidding on A, the other local bidder will signal her value for B by bidding the expected payment in case of getting the object. The global bidder gives up if that value is sufficiently high and bids the value of the global bidder otherwise.

In general, the introduction of jump bidding expands the equilibrium set, as it often happens when signalling opportunities are introduced in a dynamic game with private information. While the partition equilibrium described above would survive the introduction of jump bidding—it is enough to specify that jumps only occur out of equilibrium and that beliefs about bidders who jump-bid assign high probability on low values—the remaining question is whether the introduction of jump bidding can generate equilibria in which the exposure problem disappears.

The answer in our model is no. The crucial difference between our model and Zheng's model is that, once the bidders have signaled their values there may be cases in which the high-budget bidder is forced to buy one object at a price higher than the stand-alone value. In Zheng's model this does not happen because there are two local bidders, and the global bidder can leave the auction with zero objects. Since we have only two bidders, once the price has reached h_1 and the rich bidder will be forced to buy at least one object if he does not withdraw from both objects. Therefore, in the continuation game where the bidders jump-bid and signal their values, some types of the rich bidder may end up with a strictly negative utility. These types will thus fear exposure, and will be better off leaving the auction either before the price reaches h_1 or before any jumping occurs.

As a last point, one simple way of eliminating the exposure problem is to sell the two objects as a bundle. In this case the auction becomes a simple English auction in which bidder i has value $\min\{2(v_i + k_i), h_i\}$. In this case the bundle can go to the rich bidder when the valuation of the poor bidder is higher, because the poor bidder does not have enough money. The opposite will never happen, while it may happen in our equilibrium. Additional distortions may occur when it is efficient to split the object, i.e. $v_1 + v_2 > \max\{2(v_1 + k_1), 2(v_2 + k_2)\}$. The auction format studied in this paper allows for splitting the object, and this will sometimes result in a more efficient equilibrium outcome. Thus both the expected revenue and the expected social surplus created by the simultaneous ascending auctions can be higher or lower than an English auction for the bundle depending on the distribution of (v_i, k_i).

3.2.2 More bidders, more objects

When we add more bidders, they may or may not be budget constrained. Suppose that we add one budget–constrained bidder with a budget w_1, so that now bidders 1 and 2 are poor and bidder 3 is rich. In principle there are two effects. First, when both poor bidders reduce demand to one at price h_1, the problem for the rich bidder if she want to pursue both objects is equivalent to one in which she faces a competitor with (unknown) stand-alone value equal to $\max\{v_1, v_2\}$. Since the distribution of $\max\{v_1, v_2\}$ first-order stochastically dominates the distribution of

v_1, it becomes more likely that the rich bidder ends up with just one object. This in principle may induce a larger set of types to avoid competition.

This effect however is completely nullified by the fact that, as long as there are two bidders pursuing one object each, the rich bidder can reduce demand to zero and avoid buying one object at a price higher than the stand-alone value. Thus, when there are at least two poor bidders who demand one object each, there is no need for the rich bidder to preemptively leave the auction. In fact, the only point at which the exposure problem becomes relevant is when the next-to-last low-budget bidder reduces demand to zero, so that the rich bidder has to decide whether to leave the auction or face the remaining low-budget bidder. Note that this effect will work even if the poor bidders have different budgets, as long as both are lower than the rich bidder's budget.

Similar considerations hold if we add more unconstrained bidders. If two bidders are still active at $p = h_1$, they will compete for both objects. The exposure problem is still not completely eliminated, since the value of the poor bidder may be higher than the per-unit value of the bundle of both the rich bidders. It is clear however that this is more unlikely, hence fewer types of the rich bidders will decide to leave the auction before the price reaches their unit value. Thus adding additional rich bidders further weakens the exposure problem.

Adding more objects may or may not ameliorate the exposure problem, depending on how the bidders' preferences are extended. Suppose first that the marginal value of the third object is zero, while the valuations for one and two objects remain v_i and $2\,(v_i + k_i)$, respectively. In this case the noncollusive equilibrium is trivial: if the price reaches h_1, the objects will be immediately split, with the poor bidder getting one object and the rich bidder getting two. Second, suppose that three objects give a utility $3\,(v_i + \alpha k_i)$, with $\alpha \geq 1$ (increasing marginal utility). In this case when the price reaches h_1, the rich bidder will have to determine an optimal stopping price obtained by solving

$$\max_{p \in [h_1, h_2]} \int_{h_1}^{p} 3\,(v_2 + \alpha k_2 - v_1)\,\mathrm{d}G\,(v_1) + [1 - G\,(p)]\,2\,(v_2 + k_2 - p)\,.$$

The value of this program is always strictly positive, since $p = h_1$ is feasible and gives a strictly positive utility. Thus in this case too there is no exposure problem, i.e. the rich bidder never leaves the auction with zero objects. The only source of inefficiency is the monopsony effect.

However, it is relatively easy to reintroduce the exposure problem by changing preferences. Suppose for example that complementarities only accrue when the bidder has three objects, i.e. the utility of one object is v_i, the utility of two objects is $2v_i$ and the utility of three objects is $3\,(v_i + k_i)$. In that case the analysis would be very similar to the one in the paper, with both the exposure problem and the monopsony problem affecting the outcome.

4 Conclusions

In this paper we have analyzed the structure of noncollusive equilibria in a simultaneous ascending clock auction under the assumptions that the bidders are budget constrained and have increasing marginal values. These equilibria exhibit some intuitive properties, such as the existence of an exposure problem for the high-budget bidder. The simultaneous ascending clock auction has an efficient noncollusive equilibrium when there are no budget constraints, but it generates various inefficiencies when budget constraints are present. Not only objects are split too frequently, but it may also happen that, because of the exposure problem, the low–budget bidder may win the bundle even if her value for the bundle is lower than the value of the high budget bidder.

Appendix

Proof of Proposition 1 We specify strategies and beliefs, and show that they form a perfect Bayesian equilibrium. The equilibrium is symmetric, so the two bidders have the same strategy and beliefs.

Strategy. Type (v_i, k_i) of bidder i keeps both buttons pushed whenever the price is $p < v_i + k_i$, no matter how many buttons the opponent has previously released, and releases both buttons when the price reaches $p = v_i + k_i$. Also, the bidder releases all the remaining buttons whenever the price is $p > v_i + k_i$ (this can only happen out of equilibrium).

Beliefs. At any price p at which the opponent j has not released any button the belief on v_j and k_j is computed using the Bayes' rule, i.e. the belief on v_j is given by $F\left(v_j \middle| v_j + k_j \geq p\right)$ and the belief on k_j is given by $G\left(k_j \middle| v_j + k_j \geq p\right)$. If the opponent releases only one button at price p then the belief on v_j is any arbitrary distribution with support $[0, \min\{p, 1\}]$.

The beliefs are compatible with the strategy profile, since they are obtained using the Bayes' rule on the equilibrium path. We have to check optimality of the strategy on and off the equilibrium path.

If at any p bidder i keeps following the equilibrium strategy then the utility is

$$\max\left\{2\left[v_i + k_i - \left(v_j + k_j\right)\right], 0\right\}.$$

The only possible deviations are releasing two buttons, which gives 0, and releasing one button, which gives $\max\left\{\left[v_i - \left(v_j + k_j\right)\right], 0\right\}$. Thus, no deviation is profitable.

Out of the equilibrium path, the only case that matters is the one in which the opponent has released one button at some price $p' < p$, where p is the current price. Given the beliefs, the opponent is expected to leave immediately the auction. This implies that, by keeping both buttons pushed, the utility is $2(v_i + k_i - p)$. Releasing one or two buttons yields $(v_i - p)$, which is clearly less.

The outcome of this strategy profile is that both objects are sold to the bidder with the highest value for the two-unit bundle, say bidder i, for a price of $v_{-i} + k_{-i}$. Thus, the outcome is efficient. □

Proof of Lemma 1 Once the price arrives at h_1, bidder 1 cannot continue to bid on both objects. Reducing demand to zero is dominated by reducing demand to 1, and lifting the second button if bidder 2 does not react. Thus, in any noncollusive equilibrium, all types of bidder 1 with $v_1 + k_1 > h_1$ reduce their demand to one when the price arrives at h_1. If bidder 2 does not react, it is optimal for all types with $v_1 < h_1$ to lift the second button, as buying a single object for h_1 would generate a surplus of $v_1 - h_1 < 0$. For all types with $v_1 > h_1$ it is optimal to bid on one object up to v_1. □

Proof of Proposition 2 We have to find beliefs which are compatible with the proposed strategy profile, describe behavior off the equilibrium path and show that no profitable deviation exists.

Beliefs are similar to the ones used in the proof of Proposition 1, i.e. they are given by the Bayes' rule on the equilibrium path and they assign low values to the stand-alone type of a bidder who drops only one object at any price lower than h_1.

The only relevant out-of-equilibrium path is the one in which some bidder i reduces demand to one at a price $p < h_1$. As in Proposition 1 we prescribe that in this case the belief on v_i is any arbitrary distribution with support $[0, \min\{p, 1\}]$. Given the beliefs, it is optimal for the other bidder j to pursue both objects as long as $v_j + k_j > p$, since bidder i is expected to drop out of the auction immediately on the other object as well.

On the equilibrium path, given the definitions of the functions $v^*(k_2)$ and $p^*(v_2, k_2)$, it is obvious that the prescribed strategies are optimal. Finally, it is obvious (given the definition of optimal stopping time) that no profitable deviation exists for player 2 after bidder 1 has reduced demand to one at h_1. Also, no bidder can profit from reducing demand to one at a price $p < h_1$, since the opponent becomes more aggressive. In particular, notice that if bidder 2 reduces demand to one at $p < h_1$ then the opponent will try to get both objects until the price reaches $\min\{v_1 + k_1, h_1\}$. The outcome is therefore that bidder 2 either gets nothing or she gets only one object at a price $\min\{v_1 + k_1, h_1\}$, making the deviation unprofitable.

To show that this is the unique noncollusive equilibrium, observe that Lemma 1 pins down the behavior of the low-budget player in every non-collusive equilibrium. Given the behavior of the low-budget bidder, the unique best response of the high-budget bidder is given by the partition strategy described above. □

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