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Trans-Coordinate Hamiltonian

The wave function of a system of two *non-interacting* objects is given by

$$\Psi(\mathbf{a}, \mathbf{b}) = a(\mathbf{a})b(\mathbf{b})$$

Define the operator $\partial/\partial t_{12}$ through the trans-coordinate limiting process (Eq. 14.2 of Chap.14) acting on $\Phi(\mathbf{a}, \mathbf{b})$ to be

$$\frac{\partial}{\partial t_{12}} \Psi(\mathbf{a}, \mathbf{b}) = \frac{\partial}{\partial t_1} \Psi(\mathbf{a}, \mathbf{b}) + \frac{\partial}{\partial t_2} \Psi(\mathbf{a}, \mathbf{b}) = \frac{\partial a(\mathbf{a})}{\partial t_1} b(\mathbf{b}) + a(\mathbf{a}) \frac{\partial b(\mathbf{b})}{\partial t_2}$$

where t_1 is the time at event \mathbf{a} and t_2 is the time at event \mathbf{b} . Dropping references to \mathbf{a} and \mathbf{b} , the dynamic principle for these independent particles is therefore

$$i \frac{\partial}{\partial t_{12}} \Psi = H \Psi \quad \text{where } H = H_1 + H_2$$

$$i \frac{\partial}{\partial t_{12}} \Psi = i \frac{\partial}{\partial t_1} \Psi + i \frac{\partial}{\partial t_2} \Psi = H_1 \Psi + H_2 \Psi$$

giving

$$i \frac{\partial}{\partial t_1} \Psi = H_1 \Psi \quad i \frac{\partial}{\partial t_2} \Psi = H_2 \Psi$$

or

$$i \frac{\partial}{\partial t_1} a = H_1 a \quad i \frac{\partial}{\partial t_2} b = H_2 b$$

The expectation value of an operator $P(\mathbf{a})$ at an event \mathbf{a} in the object $a(\mathbf{a})$ evolves in the usual way.

$$\frac{\partial}{\partial t} a^* P a = \left(\frac{\partial}{\partial t} a^* \right) P a + a^* P \left(\frac{\partial}{\partial t} a \right) + a^* \frac{\partial P}{\partial t} a$$

$$\frac{\partial}{\partial t} a^* P a = i a^* [H P - P H] a + a^* \frac{\partial P}{\partial t} a$$

When P commutes with H and is explicitly time independent, its expectation values will be unchanged in time. Because the direction of the differential variable ∂t through event \mathbf{a} follows the square modular flow, the same may be said of the operator's expectation value per unit volume at event \mathbf{a} .

A quantum jump

Some interactions, like some scattering interactions, give rise to a continuous change in the wave function that occurs entirely within a single component of the wave function. Other interactions produce quantum jumps that requires the creation of new components. If for instance there is a interaction between two objects $a(\mathbf{a})$ and $b(\mathbf{b})$ giving rise to a new *ready* component $\underline{c}(\mathbf{c})$, then the state function $\Psi(\mathbf{a}, \mathbf{b}, \mathbf{c}) = a(\mathbf{a})b(\mathbf{b}) + \underline{c}(\mathbf{c})$ will be dependent on three differential times $\partial t_1, \partial t_2$, and ∂t_3 such that

$$\begin{aligned} \frac{\partial}{\partial t_{123}} \Psi(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= \frac{\partial}{\partial t_1} \Psi(\mathbf{a}, \mathbf{b}, \mathbf{c}) + \frac{\partial}{\partial t_2} \Psi(\mathbf{a}, \mathbf{b}, \mathbf{c}) + \frac{\partial}{\partial t_3} \Psi(\mathbf{a}, \mathbf{b}, \mathbf{c}) \\ &= \frac{\partial a(\mathbf{a})}{\partial t_1} b(\mathbf{b}) + a(\mathbf{a}) \frac{\partial b(\mathbf{b})}{\partial t_2} + \frac{\partial \underline{c}(\mathbf{c})}{\partial t_3} \end{aligned}$$

Again dropping references to \mathbf{a} and \mathbf{b} , the dynamic principle is now given by

$$i \frac{\partial}{\partial t_{123}} \Psi = H \Psi \quad \text{where: } H = H_1 + H_2 + \mathcal{H} \quad (15.9)$$

where \mathcal{H} is the interaction Hamiltonian. In this case the best way to proceed is in the interaction picture.

Interaction picture

The interaction picture introduces a new state variable

$$\Phi = \exp i(H_1 t_1 + H_2 t_2) \Psi$$

This function contains the integrated variables t_1 and t_2 . One can take the view that these are not coordinates because they represent an integration of dt_1 and dt_2 along the separate world lines of objects 1 and 2. However these variables do imply a choice of origins for t_1 and t_2 , and that makes them “essentially” coordinates. We therefore say that nature only goes so far as Eq. 15.9. That is entirely sufficient because that equation contains everything of importance. The function Φ is introduced as an analytic tool only. The entire interaction picture is merely analytic because it uses time coordinates.

The time derivative of Φ is given by

$$\begin{aligned} i \frac{\partial}{\partial t_{123}} \Phi &= -H_1 \Phi - H_2 \Phi + i \exp i(H_1 t_1 + H_2 t_2) \frac{\partial}{\partial t_{123}} \Psi \\ &= -H_1 \Phi - H_2 \Phi + \exp i(H_1 t_1 + H_2 t_2) (H_1 + H_2 + \mathcal{H}) \Psi \quad \text{from Eq. 15 9} \\ &= -H_1 \Phi - H_2 \Phi + (H_1 + H_2) \Phi + \exp i(H_1 t_1 + H_2 t_2) \mathcal{H} \Psi \end{aligned}$$

giving
$$i \frac{\partial}{\partial t_{123}} \Phi = \mathcal{H}_{\text{int}} \Phi$$

$$\text{where } \mathcal{H}_{\text{int}} = \exp i(H_1 t_1 + H_2 t_2) \mathcal{H} \exp -i(H_1 t_1 + H_2 t_2)$$

The new solution Φ is therefore propagated through time by the transformed interaction Hamiltonian in this picture.