Conic States

By Richard A Mould

Department of Physics and Astronomy, State University of New York,

Stony Brook, N.Y. 11794-3800. http://ms.cc.sunysb.edu/~rmould

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Abstract

It is shown that the quantum mechanical wave function of a Fermi particle can be mapped onto the surface of the backward time cone of an event in Minkowski space. That function evolves in time from one conic surface to another following a chosen world line. This evolution can be described by dynamic principle that mirrors the Schrödinger equation in the non-relativistic case, and the Dirac equation in the relativistic case. Transformations between these conic coordinates are defined in flat space, and transformations in curved space are discussed.

Introduction

It is absurd to believe that a quantum mechanical wave function will collapse upon measurement along a t = constant surface of an artificially constructed coordinate system. Only a fully covariant surface makes physical sense; and the only serious candidate is the surface of the backward time cone of the event responsible for the collapse. We call this a *conic* surface.

In this paper it is shown that the wave function of a free Fermi particle can be displayed on conic surfaces. Given a Minkowski space with only its metric intervals specified, we choose a world line through the event whose conic surface is being considered. These two things always go together in this paper as part of a single structure: (a) a backward time cone and (b) a time axis specified by t. They are shown together in Fig. 1 where the t-axis is symmetric to the conic surface.

The metric can be used to define an r-axis that accompanies this structure. It is everywhere perpendicular to the t-axis, and identifies a point on the conic surface by its proper distance from that axis. In the two dimensional Minkowski space of Fig. 1, each surface event is therefore identified by two variables: the time t_V of the vertex event and the value of r. In (3 + 1) space, $\boldsymbol{r}$ is a three-dimensional vector extending from the temporal origin to the surface of an imploding sphere that represents the surface of an incoming backward time cone.

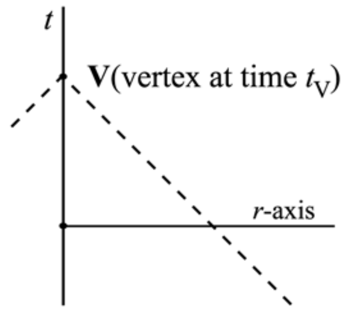

Figure 1: The basic conic structure

One must choose how this structure is oriented relative to objects in the given physical system. One choice is equivalent to selecting one Lorentz observer, and another is equivalent selecting another Lorentz observer. In the end, coordinates other than r an t_V are not used. We certainly do introduce standard space-like coordinates in order to provide a familiar beginning - in particular we add a proper s-axis along the horizontal that simulates the usual x-axis. But in the end this coordinate is removed and we are left only with the basic structure and the wave functions that have been defined on it relative to proper coordinates r and t_V alone.

A Single Free Fermion

One cycle of a plain wave $\Psi(s, t_0)$ is given along the s-axis at t_0 between s_1 and s_2 in Fig. 2, where $k(s - s_1)$ goes from 0 to 2π.

$$\Psi(s_2 \geq s \geq s_1, t \geq t_0) = A_\omega \exp i\{k(s - s_1) - \omega(t - t_0)\}$$

The function advances in time along the s-axis, occupying the shaded area in the figure.

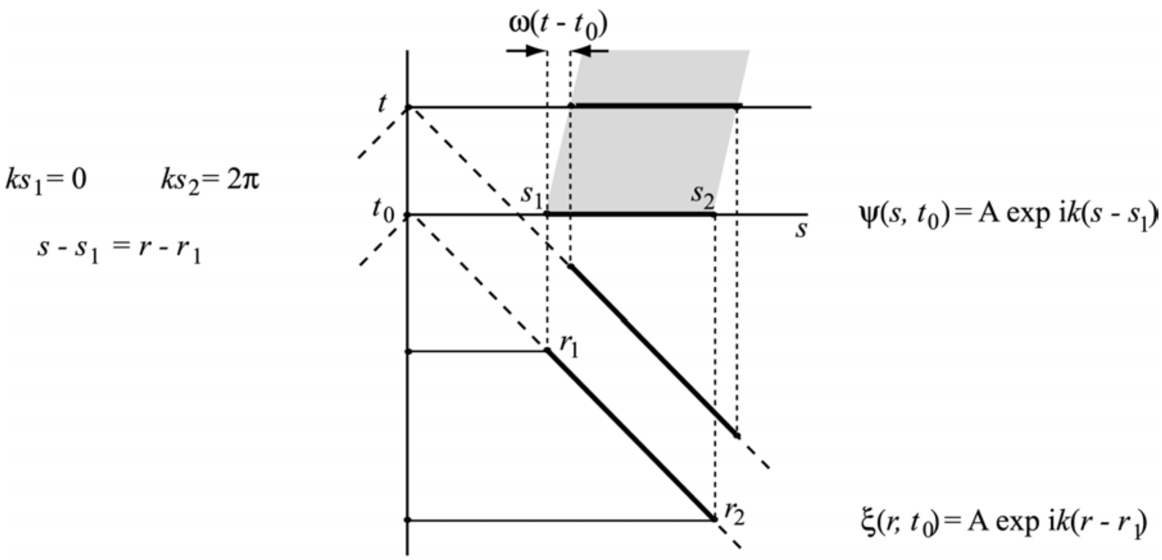

Figure 2: Mapping onto a conic surface

We map this function onto the dark line on the surface of the conic section with its vertex at t_0. Each event on that surface has an assigned coordinate r and t_0, so we define the function along the conic surface to be

$$\xi(r_2 \geq r \geq r_1, t_0) = A_\omega \exp \mathrm{i}\{k(r - r_1) - \omega t_0\} \tag{1}$$

where $s - s_1 = r - r_1$ holds for all s and r. More generally we say that a plain wave that is mapped onto a conic section with its vertex at t_V is given by

$$\xi(r, t_V) = A_\omega \exp \mathrm{i}k(r - \omega t_V)$$

specified in terms of the proper coordinates r and t_V.

The wave function in Eq. 1 can be extended to include an indefinite number of cycles. That extension can go over the vertex to the rising side of the conic surface without difficultly. Other frequencies can be added and phased to give a Fourier expansion of any desired function $\xi(r, t_V)$ on the conic surface, including a fee particle's wave packet that is confined to move within the shaded area of Fig. 2.

A Dynamic Principle

We anticipate that functions like $\xi(r, t_V)$ can be used to represent the motion of a fermion, and we want to know how to express the relevant dynamic principle. Assume that it is analogous to the non-relativistic Schrödinger equation with a dynamic operator $\hat{D}$ in place of the Hamiltonian.

$$\hat{D}\xi(r,t) = -i\hbar\partial_t \xi(r,t) \tag{2}$$

This equation is correct if we let

$$\hat{D} = \frac{h^2}{2m}\partial_r{}^2 \tag{3}$$

The function that is mapped onto successive conic surfaces is therefore propagated in time by a dynamic principle that mirrors the Schrödinger equation for a non-relativistic system. We conclude that Eqs. 2 and 3 apply in the non-relativistic case where $\xi(r, t)$ is the *conic state of a free Fermi particle*.

The same construction can be applied to each of the four components of a Dirac wave function, in which case the dynamic operator $\hat{D}$ mirrors the Dirac Hamiltonian for a free fermion. The amplitude A in the above equations is then assumed to have the required four-components.

Higher Dimensions

For a 2 + 1 space it is possible to define perpendicular axes that allow the proper distance r to be broken up into proper components r_x and r_y, so Eqs. 2 and 3 will apply to the conic wave packet along each axis. It is then possible that a (r_x, t) plane will not intersect the vertex of conic surface of interest. Figure 3 shows two such intersections.

The mapping procedure used above also applies in this case. Consider two events in Fig. 3 located on the lower conic surface (with its vertex simultaneous with t_0) that are labeled s' and s''(or r' and r''). We say that for any two such events $s'' - s' = r'' - r'$, so the s-scale maps directly onto the r-scale even though the surface is curved. The function $\psi(s, t_0)$ is mapped onto this conic surface giving $\xi(r, t_V)$ with the r-axis having the same scale as the s-axis.

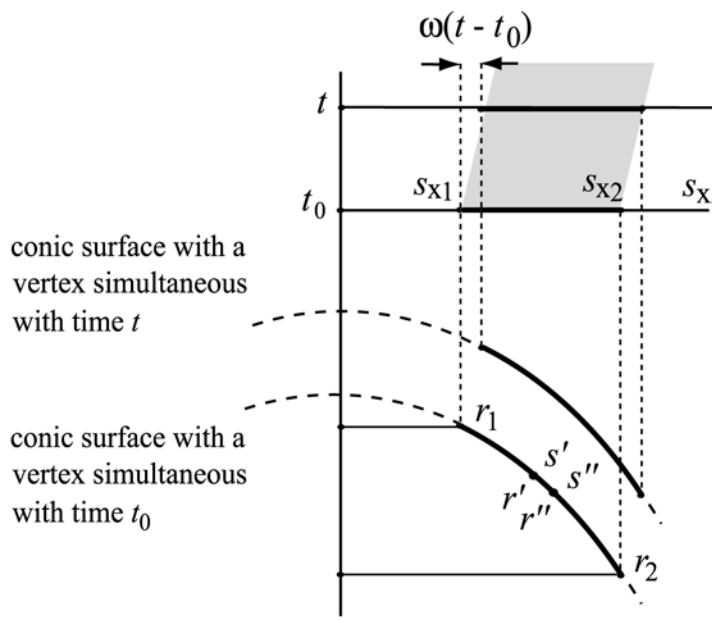

Figure 3: Off-vertex mapping

In 3 + 1 space the incoming 'spherical' conic surface will converge on a vertex time t_V. The volume that is simultaneous with t_V contains the initial condition $\psi_0(\mathbf{s}, t_v)$, and each of these values can be mapped onto the incoming spherical surface when (prior to t_v) the surface passes through that point.

The General Case

The claim is that the *s*-axis is superfluous, and that the only coordinates we need are those associated with a conic surface. If given the wave function ξ of a particle along the surface with vertex t_a in Fig. 4, the dynamic principle will carry it into all succeeding surfaces like that of t_b in that figure.

Eliminating the *s*-axis allows us to generalize to the non-free particle. As before, let the dark line on one of the conic states in Fig. 4 cover a single cycle of $\xi(r, t_V)$ from $\phi = 0$ to 2π. In that case the dynamic operator $\hat{D}$ not only advances the state $\xi(r, t_a)$ to $\xi(r, t_b)$ that occurs at a later time, but it can also change its amplitude from A_a into A_b; where again, A is understood to be the four component Dirac amplitude. Building up Fourier components of this kind into a general wave, it is clear that conic state *can represent a Fermi particle in any state of motion*.

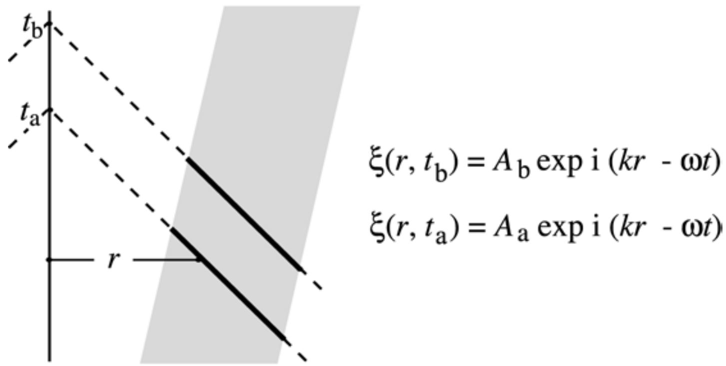

Figure 4: General conic wave

This procedure applies to Fermi particles in the system. We assume that there are also normalized fields defined on the surface of the backward time cone that provides bosons to the system as needed .

Conic Transformations

The coordinate designation (r, t_V) is satisfactory for the purpose of propagating a wave function through time with the appropriate dynamic principle. But for the purpose of coordinate transformation it is better to use (r, t), where t is defined to be the time at which r intersects the time axis in Fig. 5. The procedure is: use (r, t_V) when applying $\hat{D}$ in the original coordinates, shift to coordinates (r, t), then transform to (r', t'), and then shift back to the form (r', t'_V) when applying $\hat{D}$ in the new coordinates.

In the case of the Lorentz transformation, the transformation equations are

$$r' = \gamma[r + \beta(t - t_V)] = \gamma(r - \beta r) = \gamma r(1 - \beta)$$

$$t' - t'_V = \gamma(t - t_V + \beta r) = \gamma(-r + \beta r) = -\gamma r(1 - \beta)$$

Since the invariant interval between the event on the conic surface and the origin is equal to zero we set $r = -(t - t_V)$ in these equations. We also let $t'_V = t_V$ because a clock moving along the world line will not change its setting as a result of a coordinate transformation. These equations give the scale along each of the new axes in terms of r.

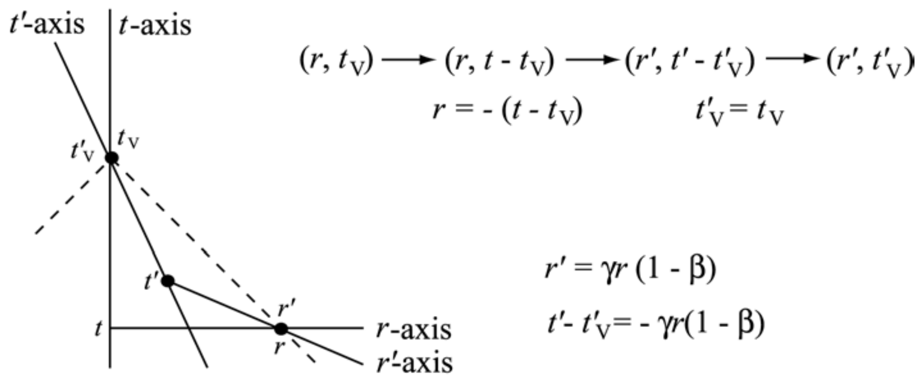

Figure 5: Lorentz transformation

Both coordinates, primed and unprimed, designate events on the conic surface, and there only. All other surfaces in the surrounding Minkowski space are coordinate free.

A similar procedure can be followed when making more general transformations in flat space. Take the space coordinate to be r and the time coordinate to be $t - t_V$. The general transformation equations are of the form

$$r'(r, t - t_V) = r'(r)$$

$$t'(r, t - t_V) - t'_V = t'(r) - t_V$$

where $t'_V = t_V$ and $r = -(t - t_V)$ for any event on the conic surface. The unprimed coordinates are Lorentzian.

This means that we can transform to any coordinates that are definable by the above transformation, including an accelerated frame or a rotation frame [1]. The new coordinates will label events on the conic surface only.

The Principle of Relativity

The guiding principle of general relativity has been that the theory must be independent of coordinates, and this is achieved by requiring the covariance of all relevant equations. Typically one chooses a set of coordinates, finds the metric tensor and then the metric. This process goes from right to left in the equation.

$$ds^2 = g_{\mu\nu}x^{\mu}x^{\nu} \tag{4}$$

Covariance is generally understood to mean that this can be done for any system of coordinates. However, this *cannot be done* in general relativity for all coordinates. It cannot be done for a system of coordinates that are discontinuous. Coordinates are used to define the metric tensor out to the second derivative, and that cannot be done with a discontinuous coordinate base. So if we say that a theory is relativistically correct only if it is possible to use *any* set of coordinates to identify events, then general relativity is not relativistically correct.

The way to understand Eq. 4 is to start from the left and go to the right. Assume that the physical situation (i.e., the distribution of matter with specified interactions) will result in a continuous metric. The problem is to find a metric tensor and associated coordinates that give that metric. Since the metric is continuous, the only workable coordinates will themselves have to be continuous. It will not matter if *not all* coordinate systems work in Eq. 4; and it will not matter *how many* coordinate systems work. The principle of relativity is fulfilled by saying that the matter distribution and interaction *gives rise to a metric that is independent of coordinates* – and that is all.

Coordinates in Curved Space

In flat space, geodesics arising from the metric will generate straight lines in the time as well as the space dimension. A spherical surface can be defined with its center at an event t_0 on the t-axis by finding the intersection of the forward time cone of another event at $t_0 - \Delta t$ and the backward time cone of $t_0 + \Delta t$. This can be done for every Δt, giving a volume as layers of spherical shells Three diameters can then by found that go through t_0 and that are perpendicular to each other, giving r_x, r_y, and r_z axes.

We cannot assume that spatial geodesics in a curved space behave as they do in flat space, so we cannot systematically proceed as we did above to construct an inclusive and single valued system of coordinates. However, we can construct an inertial frame differentially close to any event along the time axis with the radial lines clearly identified in coordinates θ and ϕ. These lines can then be projected outward along spatial geodesics[1] until they encounter the volume of intersection between the particle wave and the imploding spherical light wave of the backward light cone (in 3 + 1 space), thereby identifying each event of that volume with coordinates θ and ϕ. The *r*-value will be the proper distance to each event along the radial line. Presumably, the resulting identify-cation (r, θ, ϕ) will be inclusive and single valued for each event on the imploding conic volume in the region of interest, and the time coordinate will be the proper time $t - t_V$ of the chosen origin along the *t*-axis.

The problem then is to find a metric tensor that produces the right metric using Eq. 4. We will then have defined a set of coordinates for this general case by using only the given metric as a guide.

Objection

The main objection to the notion of conic states is based on a long-standing claim of causal inconsistency. This problem is dealt with in another paper [2].

References

1. R. A. Mould, "*Basic Relativity*" (Springer, New York, 2002), Eqs. 8.7 and 9.1

3. R. A. Mould, "*Quantum Boundaries in Relativity* ", presently on the author's web site: http://ms.cc.sunysb.edu/~rmould/index.html

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[1] We do not need the geodesic equation to define straight lines in 3-space. If given a continuous metric, three points *a*, *b*, and *c* will lie along a straight line if $ab + bc = ac$, when distances ab, bc, and ac are confined to single values.