

**POL 606 Time Series Analysis**  
**Prof. Matthew Lebo**  
**Week 10**  
**ARCH & GARCH Modeling**

**But first,**

**Encompassing Tests**

Not specific for ARCH models.

How do we tell which model we prefer?

Adjusted  $R^2$ , AIC, SBC, hypothesis tests...

Or, encompassing tests.

We can do these for *nested* models or *non-nested* models.

Two non-nested models:

$$y_t = \beta_0 + \beta_1 x_t + \varepsilon_t \quad (1)$$

$$y_t = a_0 + a_1 z_t + u_t \quad (2)$$

There is no restriction that you can place on the parameters to make one model into the other.

We can form a “joint nested” model:

$$y_t = c_0 + c_1 x_t + c_2 z_t + v_t \quad (3)$$

$t$ -test of  $c_1$  and  $c_2$  gives us some information on which we prefer.

Better to test the model as a whole rather than just the variable.

Call the models  $M_1$  and  $M_2$ .

Does  $M_2$  encompass  $M_1$ ? That is, does  $M_2 \xi M_1$ ? ( $\xi = \text{xi}$ )

Does  $M_1$  encompass  $M_2$ ? That is, does  $M_1 \xi M_2$ ?

If  $M_2$  encompasses  $M_1$  it explains all of the variance that  $M_1$  is explaining.

Davidson & MacKinnon J-test

1. Estimate  $M_2$  and calculate  $\hat{y}_{2t}$ . The 2 indicates this is for Model 2.
2. Use  $\hat{y}_{2t}$  as an additional variable in  $M_1$  so that  $M_1$  becomes:

$$y_t = \beta_0 + \beta_1 x_t + \beta_2 \hat{y}_{2t} + \varepsilon_t$$

Then we do a  $t$ -test of  $\beta_2$ .

If it is significant then  $M_2$  encompasses  $M_1$ .

We can then reverse the two and see if  $M_1$  encompasses  $M_2$ .

It is possible for both to encompass each other or for neither to encompass the other.

We can do this for 3 or more models and the results are transitive, i.e.:

If  $M_1 \xi M_2$  and  $M_2 \xi M_3$ , then  $M_1 \xi M_3$ .

In RATS:

```
*ERROR CORRECTION MODEL FOR FINANFTD
linreg consd / consdr
# constant pmsatd finanftd consr{1} falkmay falkj elect83 $
  elect87 elect92 poll event14

* ERROR CORRECTION MODEL FOR ECONFTD
linreg consd / consdt
# constant pmsatd econftd consr{1} falkmay falkj elect83 $
  elect87 elect92 poll event14

* DOES MODEL FOR ECONFTD ENCOMPASS MODEL FOR FINANFTD?
set consdel = consd - consdt
linreg consd / consdk
# constant pmsatd finanftd consr{1} falkmay falkj elect83 $
  elect87 elect92 poll event14 consdel

* DOES MODEL FOR FINANFTD ENCOMPASS MODEL FOR ECONFTD?
set consde2 = consd - consdr
linreg consd / consdkk
# constant pmsatd econftd consr{1} falkmay falkj elect83 $
  elect87 elect92 poll event14 consde2
```

## ARCH, GARCH, and Other CH Processes

Political scientists are typically interested in modeling the mean of a time series variable.

However, some analysts may be interested in studying the volatility of a series.

For example, political economists studying exchange rates.

For this purpose a class of ARCH (Autoregressive Conditional Heteroskedasticity) models pioneered by Engle (1982) are useful.

Remember that heteroskedasticity means that the error variance is not constant.

Remember that non-constant variance is a sign of non-stationarity.

However, a series can still be stationary if the *unconditional* (or long-run) variance is constant.

If there are periods in which the conditional variance fluctuates, this may not affect the long-term unconditional variance.

So we are relaxing the assumption that  $\varepsilon_t \sim N(0, \sigma^2)$ .

Now we have:  $\sigma_t^2$ .

For example, if the conditional variance follows a stationary AR(1) process, the long run variance is constant but the variance at  $t$  is dependent on the variance at  $t-1$ .

(Much of what follows are highlights from Enders Handbook).

Thus, we are interested in modeling conditional heteroskedasticity.

ARCH models express the conditional variance of stochastic errors as an AR process.

So, ARCH allows the possibility of an AR process in the error variance while GARCH allows both AR and MA processes.

GARCH (Generalized ARCH) has been modified in several ways – see Enders.

For example, ARCH-M models specify that the mean of a series is a function of its conditional variance.

Threshold ARCH (TARCH) and exponential GARCH (EGARCH) models account for asymmetries in the effects of positive and negative shocks, and integrated-GARCH (IGARCH) models consider situations of strong persistence in the conditional volatility of a series. FI-GARCH allows for the possibility of fractional integration in the variance.

ARCH models may also be extended to study how various events and conditions affect the conditional volatility of a series.

Although most political science applications of ARCH models are found in work by political economists, some analysts are now beginning to use these models to study volatility in presidential approval ratings and support for political parties.

## The Basic ARCH Model

ARCH process allows the possibility that the error term is following some kind of AR process.

Begin with a simple model of  $y_t$ :  $y_t = b_0 + b_1 y_{t-1} + \varepsilon_t$  (1)

An ARCH 1 Model:  $\varepsilon_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + u_t$  where  $\varepsilon_t^2$  is the error variance at time  $t$ . (2)

An ARCH 2 Model:  $\varepsilon_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + a_2 \varepsilon_{t-2}^2 + u_t$

More generally:  $\varepsilon_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + a_2 \varepsilon_{t-2}^2 + \dots + a_p \varepsilon_{t-p}^2 + v_t$  (3)

If there is no ARCH process at work then:  $a_1 = a_2 = \dots = a_p = 0$

Instead of estimating (1) and then (2), it would be better to estimate them simultaneously using maximum likelihood.

To do this, we should start by looking at modeling  $\varepsilon_t$  instead of  $\varepsilon_t^2$ .

We can express (2) as:  $\varepsilon_t = v_t \sqrt{a_0 + a_1 \varepsilon_{t-1}^2}$

And (3) can be expressed as:  $\varepsilon_t = v_t \sqrt{a_0 + \sum_{i=1}^q a_i \varepsilon_{t-i}^2}$  (4)

Where  $v_t$  is a white noise process with variance,  $\sigma_v^2 = 1$ ;  $v_t$  and  $\varepsilon_{t-1}$  are independent;  $a_0$  and  $a_i$  are constants with  $a_0 > 0$ , and  $0 < a_i < 1$ .

In (4) all shocks from  $\varepsilon_{t-1}$  to  $\varepsilon_{t-q}$  have a direct effect on  $\varepsilon_t$  so that the conditional variance acts like an autoregressive process of order  $q$ .

ARCH models can appear in a number of contexts since the residuals from (1) we are modeling can come from any type of time series model we have discussed so far.

These are best estimated with maximum likelihood.

These models can be done in RATS which now has window commands to help. E-Views also automates the process with menus.

## GARCH Models

We can also think of errors as being ARMA process which Generalizes the process to a GARCH model (Bollerslev, 1986).

$$\varepsilon_t^2 = h_t$$

$$\varepsilon_t = v_t \sqrt{h_t} \quad \text{where} \quad h_t = a_0 + \sum_{i=1}^q a_i \varepsilon_{t-i}^2 + \sum_{i=1}^p \beta_i h_{t-i} \quad (5)$$

$v_t$  is a white noise process.

So the conditional and unconditional means of  $\varepsilon_t$  are zero.

$$E(\varepsilon_t) = E(v_t \sqrt{h_t}) = 0$$

The main point is that the conditional variance of  $\varepsilon_t$  is given by  $E_{t-1}(\varepsilon_t^2) = h_t$ .

Thus, the conditional variance of  $\varepsilon_t$  is the ARMA process given by (5).

Using (5) we can refer to ARCH( $p,q$ ) models.

If  $q=0$ , then all  $\beta_i$  in (5) are equal to zero and we have an ARCH process.

For example, if the GARCH process is an ARMA (1,1):

$$h_t = a_0 + a_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}$$

Remembering from our fitting of AR and MA parameters in traditional Box-Jenkins, we can approximate MA processes using AR parameters.

But it may be more parsimonious to be able to specify both AR and MA processes.

So, instead of using ARCH and having additional AR components to approximate the MA process, the GARCH model may allow a more parsimonious model of the volatility.

The key to ARCH and GARCH is that the conditional variance of the disturbances of  $y_t$  constitute an ARMA process.

Its important to study the variance of the disturbances, not just the mean.

That is, the second moment matters, not just the first.

You can make a good model of  $y_t$  and have white noise residuals evident in ACF and PACF, but looking at the squares of the residuals may indicate an ARCH or GARCH process.

### Testing for ARCH and GARCH

To begin to test for presence of ARCH or GARCH processes, look at residuals and squared residuals. We are looking for an AR process.

To see if an ARCH process is at work, we perform Engle's Lagrange Multiplier Test.

Note that this test was specifically designed for ARCH but that it works also in identifying GARCH processes.

Think about:  $\varepsilon_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\varepsilon_{t-2}^2 + \dots + a_p\varepsilon_{t-p}^2 + v_t$

If the  $a_i \neq 0$  then there is information in the squared residuals.

$H_0$ : (No ARCH or GARCH)  $a_1 = a_2 = \dots = a_p = 0$

$H_1$ : (ARCH or GARCH) at least one  $a_i \neq 0$

Step one: Compute squared residuals from mean equation regression.

Step two: Estimate Auxiliary regression where we regress squared residuals on themselves @  $p$  lags.

$\varepsilon_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\varepsilon_{t-2}^2 + \dots + a_p\varepsilon_{t-p}^2 + v_t$  where  $v_t$  is the error.

Step 3: Form the LM test statistics:  $R^2 * T$ .

$R^2$  is from the auxiliary regression.

T is the sample size of the auxiliary regression.

This is distributed as  $\chi^2$  with  $p$  degrees of freedom.

The null hypothesis is there is no ARCH process.

Page 57 of Enders Handbook gives an additional test using the Ljung-Box Q-Statistic to help identify GARCH errors.

### ARCH-M Model

The mean of some process ( $y_t$ ) is a function of the conditional variance ( $h_t$ ) of that process.

$$y_t = \mu_t + \varepsilon_t$$

$$\mu_t = \beta_0 + \beta_1 h_t \text{ where } \beta_1 > 0, h_t \text{ is an ARCH}(q) \text{ process: } h_t = a_0 + \sum_{i=1}^q a_i \varepsilon_{t-i}^2.$$

### Asymmetric GARCH

Glosten *et al.* (1993) demonstrate that the expected variance in the next period may depend on whether the last change in this period was positive or negative.

For example, here is a GARCH (2,1) model with asymmetric effects:

$$h_t = c_0 + a_1 u_{t-1}^2 + b_1 h_{t-1} + b_2 h_{t-2} + m_1 u_{t-1}^2 I_{u>0}$$

where I is an indicator function such that it equals 1 when residuals of the series are positive and equals 0 otherwise.

A negative value of  $m$  means that periods with negative residuals will be immediately followed by periods of higher variance than will be periods of positive residuals.

This is exactly what one would expect for the approval series where unanticipated drops signal periods of greater volatility.

Here are some RATS results for the Presidential approval series. “D” is the asymmetric parameter.

```
GARCH Model - Estimation by BFGS
Convergence in 25 Iterations. Final criterion was 0.0000076 < 0.0000100
Monthly Data From 1978:02 To 2004:07
Usable Observations 318
Function Value -895.75439279
```

Variable	Coeff	Std Error	T-Stat	Signif
1. Mean	0.004609122	0.204688494	0.02252	0.98203496
2. C	1.516122098	0.659166323	2.30006	0.02144482
3. A	0.422870595	0.041667725	10.14864	0.00000000
4. B{1}	0.272489513	0.133861254	2.03561	0.04178938
5. B{2}	0.420825297	0.124733362	3.37380	0.00074138
6. D	-0.304912201	0.062457356	-4.88193	0.00000105