Drawing from distributions using the Probability Integral Transform: the lognormal case.

- When dealing with a dynamic or static model under uncertainty we will often have to solve an integration problem of a distribution from which it might be difficult or slow to draw.
- Think of the problem of calculating:

$$\int_r V(r)\ fr\ dr, \tag{1}$$

which after a change of variables can be written as

$$\int_0^1 V(F^{-1})\ du, \tag{2}$$

which we are going to approximate with the Probability Integral Transform. Then as we saw in previous lecture notes we can compute the integral with Gaussian Quadrature as

$$\sum_{i=1}^{N} w_i\, V(F^{-1}(u_i)), \tag{3}$$

where w_i are the quadrature weights, and u_i are the quadrature abscissae.

- Now say the function we are trying to integrate out follows a given distribution, for example the lognormal. This lognormal has mean μ and standard deviation σ.

$$Pr\{(\tilde{z}) \leq x\} = Pr\{\ln(\tilde{z}) \leq \ln(x)\} \tag{4}$$

and

$$Pr\{\frac{\ln(\tilde{z}) - \mu}{\sigma} \leq \frac{\ln(x) - \mu}{\sigma}\} \tag{5}$$

from here

$$x = \exp(\mu + \sigma\,\Phi^{-1}(u)) \tag{6}$$

where u can be a uniform random number when using the Probability Integral Transform as explained above and below.

- The use of the Probability Integral Transform follows from the following theorem, you can see the proof in Casella and Berger (1990):

Let X have continuous c.d.f $F_x(x)$ and define the random variable Y as $Y = F_x(x)$. Then Y is uniformly distributed on (0,1), that is, $P(Y \leq y) = y,\ 0 < y < 1$.

- This is used in generating random samples from a particular distribution. If it is required to generate an observation x from a population with c.d.f. F_x, we only need to generate a random number u, between 0 and 1, and solve for x in the equation $F_x(x) = u$. This has general applicability.