Polynomial Approximation

- Polynomial approximation is a special case of polynomial series approximation which in turn is a special case of the class of linear approximation algorithms.
- These algorithms approximate the time t value function $V_t(s)$ by a linear combination of the first k elements of an infinite sequence of polynomial basis functions $(\rho_1(s), \rho_2(s), \cdots, \rho_k(s), \cdots)$ defined over all $s \in S$.
- By specifying a particular set of coefficients $\hat{\theta}_t = (\hat{\theta}_{t,1}, \cdots, \hat{\theta}_{t,k})$ we obtain an estimate $V_{\hat{\theta}_t}$ of the time t value function V_t given by

$$V_{\hat{\theta}_t}(s) = \sum_{i=1}^{k} \hat{\theta}_{i,t}\, \rho_i(s). \tag{1}$$

- To implement this method we need to specify the basis function and a method for choosing the "best fitting" coefficient vector $\hat{\theta}_t$.
- A natural choice for the set of basis functions are the "ordinary" polynomial $\rho_i(s) = s^i$ since the Weierstrass approximation theorem shows that if S is compact, then the set of all linear combinations of the form (1) is dense in the space $C(S)$ of all continuous functions on S.
- A natural way to determine the $\hat{\theta}_t$ is with the nonlinear least square approach we already described.
- One problem with this estimation using ordinary polynomial basis $(1, s, s^2, \cdots)$ is that successive terms in the series approximation (1) become highly collinear as k gets large. And this can create problems in the least square estimation of $\hat{\theta}_t$.
- For example if the grid points are not well distributed it is not rare to see explosive oscillations of the approximate value function as k grows large.
- This motivates the use of uniformly bounded, orthogonal polynomial series approximations. There are (infinitely) many such orthogonal bases for $C(S)$.
- Chebyshev polynomials (p_i) are an orthogonal collection of polynomials defined on the domain $S = [-1, 1]$ where orthogonality is defined using the weighting function $\eta(ds) = ds/\sqrt{1 - s^2}$.

- The explicit formula for (p_i) is given by $p_i(s) = \cos(i\cos^{-1}(s))$. But they can also be defined by the recursion

$$p_i(s) = 2\, s\, p_{i-1}(s) - p_{i-2}(s), \tag{2}$$

with initial conditions $p_0(s) = 1$ and $p_1(s) = s$.

- By definition the Chebyshev polynomials satisfy a continuous orthogonality relation, and the more important discrete orthogonality relation

$$\sum_{l=1}^{k} p_i(s_l^k) p_j(s_l^k) = 0, \; for\; i \neq j\;, \tag{3}$$

where s_l^k are the zeros of p_k given by

$$s_l^k = \cos\left(\frac{(2l-1)\pi}{2k}\right), l = 1, \cdots, k. \tag{4}$$

This result implies that the Chebyshev zeros $(s_l^k, \cdots, s_k^k)$ can serve as a set of grid points for interpolation of an arbitrary continuous function f.

- Using these grid points we can interpolate an arbitrary continuous function f by the function $\hat{f}_{k-1}(s)$ defined by:

$$\hat{f}_{k-1}(s) = -\frac{1}{2}\hat{\theta}_1 + \sum_{i=1}^{k-1} \hat{\theta}_{i+1} p_i(s), \tag{5}$$

where the coefficients $\hat{\theta} = (\hat{\theta}_1, \cdots, \hat{\theta}_k)$ are given by:

$$\hat{\theta}_t = \frac{2}{k} \sum_{l=1}^{k} f(s_l^k) p_{i-1}(s_l^k). \tag{6}$$

- The amount of work involved in computing the coefficient $\hat{\theta}_t$ at each stage t using the $k-1$ Chebyshev interpolant is of order k^2. Once these coefficients are computed the effort required to evaluate $\hat{f}_k$ at any $s \in S$ is only of order k.

- The smoothness properties of Chebyshev polynomials lead to the presumption that we can obtain good approximations for small values of k.

Smooth Approximation of the Decision Rule

- In many problems the primary interest is in approximating the optimal decision rule α. Given an estimate $\hat{V}$ of V we can compute the implied estimate $\hat{\alpha}(s) = (\hat{\alpha}_0(s), \cdots, \hat{\alpha}_T(s))$ at any point $s \in S$ via the dynamic programming recursion:

$$\hat{\alpha}_t(s) = arg \max_{a \in A(s)} \left[u(s,a) + \beta \int \hat{V}_{t+1}(s') p(ds'|s,a) \right], t = 0, \cdots, T-1. \quad (7)$$

- In practice $\hat{\alpha}_t(s)$ will differ from $\alpha_t(s)$ not only because $\hat{V}_t$ differs from V, but also due to any additional approximation errors introduced by numerical solution of the integration and constrained maximization subproblems in the equation above.

- If we need to evaluate $\hat{\alpha}$ at a large number of points in S, say for purposes of stochastic simulation of the optimal decision rule, it will generally be too time-consuming to compute $\hat{\alpha}_t(s)$ at many different points $s \in S$.

- Then it makes sense to incur in the up-front cost of using one of the function approximation methods available (polynomial approximation, Chebyshev polynomial approximation...) to compute an approximate decision rule $\hat{\hat{\alpha}}_t$ given the information $I_N = (\hat{\alpha}(s_1), \cdots, \hat{\alpha}(s_N))$ on the optimal decisions computed over a finite grid of points $(s_1, \cdots, s_N)$ in S.

- This results in a smooth approximation $\hat{\hat{\alpha}}_t$ to the optimal decision rule α that can be quickly evaluated at any $s \in S$.