Stochastic Growth Model

- The Stochastic Growth Model has been one of the workhorses of the advancement of numerical methods in economics. It is always difficult to compare techniques and methodologies and finding a non-trivial problem that can be used to compare results is important.

- However, we have to realize the generality of the techniques we are using. This very short presentations will try to help you understand the Matlab code that uses Value Function Iteration, and then Policy Function Iteration to solve the problem.

- The problem is that of

$$\max_{c_t, k_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \tag{1}$$

  subject to

$$\begin{aligned} c_t + i_t &= z_t A k_t^{\alpha} && (2) \\ k_{t+1} &= (1-\delta) k_t + i_t && (3) \end{aligned}$$

  and z_t is a Markov chain of the technology shock to capital. Where δ is a depreciation rate, i_t is current period investment, and $\alpha \in [0,1]$.

- You can see that the problem in itself is simple, and a good test for our methods. We will find as expected that policy function iteration is amazingly faster than value function iteration, and they give the same results.

- You can find broader discussions of this model and methods to solve it in Santos (1999) chapter in the Handbook of Macroeconomics.

Deterministic Dynamic Programming

- Almost every economics problem out there involves at some level solving a constrainted maximization problem. In this class we are discussing how to solve stochastic dynamic constrained maximization problems.
- Last semester Stefan Krieger discussed how you can solve these using stochastic Lagrange multipliers and linearization methods. And these techniques work for a wide range of models. They are relatively easy to apply and, in general, linear systems can be solved relatively easily and quickly on a computer. So these methods are great to have in your toolbox.
- But these types of methods work best on smooth problems. That is one problems in which linerizing the first order conditions around some point is a resonable good approximation in the range you are interested in to the true non-linear equation.
- But not all problems in economics are smooth. In many problems you have constraints with kinks in them.
  - borrowing constraints
  - investment may have to be non-negative
- A lot of problems in economics have discrete choices, not continuous,
  - A firm may choose to enter or not enter a market
  - A firm may wish to open a second shift or not.
  - A person may choose to retire or not
  - A person may choose to get preganant or not.
- But it turns out that if we can write the problem we are interested in the following form:

$$\max_{\{u_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \tag{1}$$

subject to

$$x_{t+1} = g(x_t, u_t)$$
$$x_0 \text{ given}$$

This is the notation from Ljungvist-Sargent; x_t is the state, u_t is the control. r is the one period return function. LOM for the state.

If we can write the problem in this form, we can use some methods that handle kinks and discrete choices well. Note that we can use linearization to solve these models. But we are going to focus on some others methods that solve models of this form that can handle kinks and disceter choices well.

- There are of course down sides to setting up a problem as a dyanmic programming problem.
  - It can be quite awkward to bang your problem into this form.
  - The methods I am going to show you work exactly for only a few special cases. All other case require you to make some sort of approximation which could be worse than linearization.
  - The methods I am going to discuss in genreal are very computationally demanding. They take a lot of computer memory and they take a lot of computer time to solve the model.
- Chris discussed the general case. Today I want to present dyanmic programming within the framework on a single simple model.

1 The deterministic one-sector growth model

- no leisure $L_t = 1$ (marginal utility of leisure set to 0)
- complete depreciation, $\delta = 1$
- no uncertainty $A(t) = A \quad \forall t$.

1.1 finite horizon case

$$\max_{\{c_t, k_{t+1}\}_0^T} \sum_{t=0}^{T} \beta^t U(c_t)$$

subject to

$$c_t + k_{t+1} = Af(k_t)$$

$$k_0 \text{ given}$$

$$k_t \geq 0$$

- substituting the constraint into the utility function yields:

$$\max_{\{k_{t+1}\}_0^T} \sum_{t=0}^{T} \beta^t U(Af(k_t) - k_{t+1})$$

given the initial conditions stated above.

Could take a second-order Taylor expansion here, and solve this Kydland-Prescott style.

Taking first-order conditions yields

$$\beta^t U'(Af(k_t) - k_{t+1})(-1) + \beta^{t+1} U'(Af(k_{t+1}) - k_{t+2}) Af'(k_{t+1}) = 0$$

- holds for $t = 0, 1, 2, ...T - 1$
- at time T, $k_{t+1} = 0$
- T non-linear equations and T unknowns
- second-order difference equation (this because of the special case we picked; it could be a lot worse.)
- same equation at each date t with the exception of T
- tough to solve
- we could put this into a nonlinear equation solver. Or we could linearize the system and solve it using the mehtods you learned from Chris.

1.2 Pick functional forms

- $U(c_t) = \ln c_t$
- $Af(k_t) = Ak^\alpha$.
- The first order condition then becomes

$$\begin{aligned} \beta^t \frac{1}{Ak_t^\alpha - k_{t+1}}(-1) + \beta^{t+1} \frac{\alpha A k_{t+1}^{\alpha-1}}{Ak_{t+1}^\alpha - k_{t+2}} &= 0 \quad \text{for t=0,1,2, ...T-1} \\ k_{T+1} &= 0 \end{aligned}$$

- Still have T non-linear equations and T unknowns.

Trick – change the variables Define

$$X_t = \frac{k_{t+1}}{Ak_t^\alpha} \tag{2}$$

so X_t is the savings rate. Substituting back into the first order condition yields

$$X_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{X_t} \tag{3}$$

We know $X_T = 0$, so we can solve (3) by backwards substitution, then use k_0 to get the capital stock. Working backwards yields:

$$X_t = \alpha\beta \frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t+2}} \tag{4}$$

2 Infinite horizon case

For the infinite horizon case, we simply rewrite the problem as:

$$\max_{\{k_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^\alpha - k_{t+1}) \tag{5}$$

but we lose the boundary condition: $X_T = 0$, and we have to instead think about transversality conditions. We know from previous lectures the transversality condition for this problem takes the form of

$$\lim_{T\to\infty} \beta^T \left(\frac{\alpha A k_T^{\alpha-1}}{Ak_T^\alpha - k_{T+1}} \right) k_T = 0$$

- If we take first-order conditions, we get an infinite number of non-linear second-order difference equations and infinite number of unknowns.
- That's tough to solve.
- But note that in the finite time model that if T is large

  $$X_t \approx \alpha\beta$$

  or

  $$k_{t+1} \approx \alpha\beta A k_t^\alpha$$

So let's think for a moment.

- The planning problem takes the same form every period

- the only thing that changes each period is the capital stock

This suggests a conjecture: the solution to this infinite horizon problem has the form

$$k_{t+1} = h(k_t), \quad t = 0, 1, ...$$

In particular

$$k_{t+1} = \alpha\beta A k_t^{\alpha}.$$

- We are conjecturing that the planner's choice of next period's capital stock appear to be solely a function of the current period's capital stock and the parameters of the model.

- The function h is going to be called a *policy function*, and in particular our conjecture is that h is *time invariant.*

So this suggests another approach.

- Instead of choosing an infinite sequence of $\{c_t, k_{t+1}\}$, let's think about the social planner's problem at date 0. Really the planner only has to choose today's consumption and tomorrow's capital stock each period. And we are conjecturing that the only thing that matters in this decision is today's capital stock.

- So suppose that the infinite period problem had already been solved for all possible values of k_0. Then we could define a function

$$V(k_0) = \max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^{\alpha} - k_{t+1}).$$

  So the function $V(k_0)$ expresses the optimal value of the original problem, starting at the initial condition k_0. In other words, $V(k_0)$ is the highest discounted stream of utilities from period zero on that could be obtained with a beginning of the period capital stock k_0.

- We can then go forward one period and define:

$$V(k_1) = \max_{k_{t+1}} \sum_{t=1}^{\infty} \beta^{t-1} \ln(Ak_t^{\alpha} - k_{t+1}).$$

- Given this function $v(k_1)$, the planner's problem in period 0 would be:

$$\max_{k_1} [\ln(Ak_0^{\alpha} - k_1) + \beta V(k_1)].$$

If the function $V(k_1)$ were known, we could solve this problem for the policy function h such that:

$$k_1 = h(k_0) \quad \text{and}$$

$$c_0 = Ak_0^\alpha - h(k_0).$$

But note that we have defined $V(k_0)$ to be the maximized value of the original problem. Thus $V(k_0)$ must be the maximized value of the on period zero problem as written above:

$$V(k_0) = \max_{k_1}[\ln(Ak_0^\alpha - k_1) + \beta V(k_1)].$$

- Now we have the problem in a recursive structure.
- If we go out t+1 periods we get

$$V(k_t) = \max_{k_{t+1}}[\ln(Ak_t^\alpha - k_{t+1}) + \beta V(k_{t+1})].$$

Furthermore, we can just drop the subscripts:

$$V(k) = \max_{k'}[\ln(Ak^\alpha - k') + \beta V(k')],$$

where the prime denotes next period's value. So instead of finding an infinite sequence of $\{k_{t+1}\}$ that maximizes (5) by solving an infinite number of non-linear equations, we are finding a policy function, $h(k)$, and a value function, $V(k)$, that solve a continuum of maximization problems. So what we have done is break a single large dimensional optimization problem into a sequence of much smaller optimization problems.

The solution to this Bellman equation is:

$$V(k) = \ln(Ak^\alpha - h(k)) + \beta V(h(k))$$

This equation is a *functional equation* and can be solved for the pair of unknown functions $V(k)$ and $h(k)$.

3 The general deterministic control problem

return to the genreal case...

$$\max_{\{u_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \tag{6}$$

subject to

$$x_{t+1} = g(x_t, u_t)$$
$$x_0 \text{ given}$$

- In the one-sector growth model above, k_t was the state variable, and c_t was the control variable.

- We define
$$V^*(x_0) \equiv \max_{\{u_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \text{ subject to } x_{t+1} = g(x_t, u_t)$$
for a given x_0.

So in the general set-up we write down the Bellman equation as:

$$V(x) = \max_u \{r(x, u) + \beta V(x')\} \tag{7}$$

subject to:

$$x' = g(x, u)$$

4 Properties of the Bellman Equation

Since I have only three lectures to go over dynamic programming and show you how to write programs on the computer, I am going to follow Sargent's lead and show you a bunch of results without going over the proofs. I note where the proofs are if you wish to look at them yourself.

Proofs are in Sims notes, Stokey-Lucas-Prescott Most standard discussions of dynamic programming follow the work of Blackwell (1965) and assume the following:

1. r is concave and bounded, and

2. the set $\{(x_{t+1}, x_t) : x_{t+1} \in g(x_t, u_t), u_t \in R^k\}$ is convex and compact.

Then we can show:

1. $V(x)$ is a monotonically increasing function (Stokey-Lucas-Prescott Theorem 4.7).

2. $V(x)$ is strictly concave and $h(k)$ is a continuous single-valued function (SLP Theorem 4.8).

3. (*The Principle of Optimality*) The solution to (7) is $V^*(x_0)$.

4. This solution is approached in the limit as $j \to \infty$ by iterations on:

$$V_{j+1}(x) = \max_{u} \{r(x,u) + \beta V_j(x')\}$$

subject to

$$x' \in g(x,u)$$

Note we are iterating "backwards." Sims has a proof of this in his lecture notes.

5. (*Benveniste and Scheinkman*) The limiting value function V is differentiable with

$$V'(x) = \frac{\partial R}{\partial x}[x, h(x)] + \beta \frac{\partial g}{\partial x}[x, h(x)] V'(g[x, h(x)])$$

This is Theorem 4.10 in SLP. This is the envelop condition

For many interesting models in macroeconomics $r(x,u)$ is not bounded. Indeed, in the one-sector growth model above, $r(x,u) = \ln c$, and is thus an unbounded function. There is some discussion of this in Stokey-Lucas-Prescott. Fernando Alvarez and nancy Stokey have a recent paper where they provide conditions for these 5 results to hold with unbounded returns.

Note that if we go back to our special case of the one sector growth model and apply Benveniste and Scheinkman, we get:

$$V'(k) = \frac{\alpha A k^{\alpha-1}}{Ak^{\alpha} - k'}.$$

Take the FOC for the right-hand side of (7) yields:

$$0 = \frac{-1}{Ak^{\alpha} - k'} + \beta V'(k')$$

These two derivatives imply:

$$\frac{1}{Ak^{\alpha} - k'}(-1) + \beta \frac{\alpha A k'^{\alpha-1}}{Ak'^{\alpha} - k''} = 0$$

This formula should look familiar. Both ways of writing down the problem yields identical Euler equations.

5 Solving the Bellman equation

5.1 Value function iteration (working backwards)

We are looking for a fixed point of a concave functional equation. So we can:

1. start off with a bounded and continuous initial $V_0(x)$.

2. Solve the one period problem

$$V_1(x) = \max_u \{r(x,u) + \beta V_0(x')\}$$

subject to

$$x' \in g(x,u)$$

3. Take the value $V_1(x)$ that solves the above maximization problem and solve

$$V_2(x) = \max_u \{r(x,u) + \beta V_1(x')\}$$

subject to

$$x' \in g(x,u)$$

4. Repeat until the $h(x_t)$ "stops changing" and/or $V_j(x)$ and $V_{j+1}(x)$ are "close."

This algorithm is straight-forward, and it works. But it tends to be quite slow.

5.2 Back to the OSGM

The one-sector growth model with the functional forms we have chosen is a nice case since we can solve the problem by hand using a variety of methods. Consequently it is often used as an example when researcher are evaluating solution techniques.

1. Choose $V_0(k) = 0$.

2. Set $V_1(k) = \max_{k'} \{\ln(Ak^\alpha - k') + \beta 0\}$

3. The solution is to set $k' = 0$ so $V_1(k) = \ln(Ak^\alpha) = \ln A + \alpha \ln k$.

4. Set $V_2(k) = \max_{k'} \{\ln(Ak^\alpha - k') + \beta(\ln A + \alpha \ln k')\}$

5. The solution is to set $k' = \frac{\beta\alpha}{1+\beta\alpha}Ak^\alpha$ so

$$\begin{aligned} V_2(k) &= \ln\left(Ak^\alpha - \frac{\beta\alpha}{1+\beta\alpha}Ak^\alpha\right) + \beta\left(\ln A + \alpha \ln\left(\frac{\beta\alpha}{1+\beta\alpha}Ak^\alpha\right)\right) \\ &= \ln\left(\frac{A}{1+\beta\alpha}\right) + \beta \ln A + \alpha\beta \ln\left(\frac{\beta\alpha A}{1+\beta\alpha}\right) + (\alpha + \alpha^2\beta)\ln k \end{aligned}$$

6. repeat enough times until you see that the value function follows a geometric sequence that converges to

$$V(k) \quad = \quad = \frac{1}{1-\beta}\left[\ln(A(1-\beta\alpha)) + \frac{\beta\alpha}{1-\beta\alpha}\ln(A\beta\alpha)\right] + \frac{\alpha}{1-\alpha\beta}\ln k.$$

The associated policy function is

$$k' \quad = \quad \beta\alpha A k^{\alpha}.$$

5.3 Guess and Verify

- rich skit show fodder
- Guess a functional form for $V(k)$.
- Verify the functional form of type guessed solves the Bellman equation and deduce the values for the coefficients.

There are only a handful of models for which you can use this method. How do you form a good guess? Rule of Thumb:

$V(X)$ = constant+coefficient×the state variable in the same function form as the one-period return function

5.4 Back to the OSGM II

1. Guess $V(k) = E + F\ln(k)$.

2. Now verify

$$E + F\ln(k) = \max_{k'}\left\{\ln(Ak^{\alpha} - k') + \beta(E + F\ln(k'))\right\}$$

- Take the first-order condition

$$0 = \frac{-1}{Ak^{\alpha} - k'} + \beta F\frac{1}{k'}$$

Solving for k'

$$k' = \frac{\beta F}{1+\beta F}Ak^{\alpha}$$

- Substitute back in to the Bellman equation

$$\begin{aligned} E + F\ln k &= \ln\left(Ak^{\alpha} - \frac{\beta F}{1+\beta F}Ak^{\alpha}\right) + \beta\left(E + F\ln\left(\frac{\beta F}{1+\beta F}Ak^{\alpha}\right)\right) \\ &= \ln\left(\frac{A}{1+\beta F}\right) + \alpha\ln k + \beta E + \beta F\ln\left(\frac{\beta F A}{1+\beta F}\right) + \beta F\alpha\ln k. \end{aligned}$$

- equate coefficients on the ln terms

$$F \ln k = \alpha \ln k + \beta F \alpha \ln k$$

$$F = \frac{\alpha}{1 - \beta\alpha} \tag{8}$$

- equate coefficients on the constant term

$$E = \ln\left(\frac{A}{1+\beta F}\right) + \beta E + \beta F \ln\left(\frac{\beta A F}{1+\beta F}\right)$$

- use equation (8) to substitute out F
- do a lot of algebra, and solve for E

5.5 Howard's Policy Improvement Algorithm

Also called policy-function iteration since you iterate on the policy function instead of the Bellman equation. This routine is often much faster than value-function iteration.

1. Pick a feasible policy, $u = h_0(x_0)$, and compute the value of sticking with that policy forever:

$$V_{h_0}(x_0) = \sum_{t=0}^{\infty} \beta^t r(x_t, h_0(x_t)), \quad \text{where} \quad x_{t+1} = g(x_t, h_0(x_t)).$$

2. Choose a new policy function $h_1(x)$ that maximizes the following two period problem:

$$\max_u \{r(x,u) + \beta V_{h_0}(g(x, h_0(x)))\}$$

3. repeat steps 1 and 2 with the updated policy function until the policy function "stops changing."

5.6 Back to OSGM III

1. Pick feasible policy function $k_{t+1} = h_0(k_t) = \frac{1}{2} A k_t^\alpha$
2. Compute

$$\begin{aligned} V_{h_0}(k_0) &= \sum_{t=0}^{\infty} \beta^t \ln\left(A k_t^\alpha - \frac{1}{2} A k_t^\alpha\right) \\ &= \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{1}{2} A k_t^\alpha\right) \\ &= \sum_{t=0}^{\infty} \beta^t \left(\ln\left(\frac{1}{2}A\right) + \alpha \ln k_t\right) \end{aligned}$$

Note

$$\begin{aligned} k_t &= \frac{1}{2}Ak_{t-1}^{\alpha} \\ &= \frac{1}{2}A\left[\frac{1}{2}Ak_{t-2}^{\alpha}\right]^{\alpha} \\ &= \left(\frac{1}{2}\right)^{1+\alpha} A^{1+\alpha}k_{t-2}^{\alpha^2} \end{aligned}$$

repeated recursive substitution yields:

$$k_t = Dk_0^{\alpha^t}$$

where D denotes a constant term so,

$$\ln k_t = \ln D + \alpha^t k_0.$$

Therefore

$$\begin{aligned} V_{h_0}(k_0) &= \sum_{t=0}^{\infty}\beta^t\left(\ln(\frac{1}{2}A) + \alpha\ln D + \alpha^{t+1}\ln k_0\right) \\ &= \text{constant term} + \frac{\alpha}{1-\beta\alpha}\ln k_0 \end{aligned}$$

so

$$V_{h_0}(k') = \text{constant term} + \frac{\alpha}{1-\beta\alpha}\ln k'.$$

3. Evaluate the two period problem:

$$\max_{k'}\left\{\ln(Ak^{\alpha}-k') + \beta\left[\text{constant term} + \frac{\alpha}{1-\beta\alpha}\ln k'\right]\right\}.$$

Taking the first-order condition yields:

$$\frac{-1}{Ak^{\alpha}-k'} + \frac{\beta\alpha}{1-\beta\alpha}\frac{1}{k'} = 0.$$

Thus

$$k' = \alpha\beta Ak^{\alpha}.$$

The Howard improvement algorithm converges in a single step!