Finite Horizon Consumption/Saving problem with Capital Uncertainty

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In this note we derive the closed form solution of the finite horizon version of Phelps (1962) consumption/saving problem assuming a CRRA utility function. Our derivation is also close in nature to the one performed in Levhari and Srinivasan (1969). We can again solve this problem relying on Dynamic Programming and Bellman's principle of optimality, using backward induction. In the last period of life agents solve

$$V_T(w) \quad = \quad \max_{0\leq c\leq w} \frac{c^{1-\gamma}}{1-\gamma} + K\,\frac{(w-c)^{1-\gamma}}{1-\gamma}\,,$$

where γ is the coefficient of relative risk aversion and K is the bequest factor, characterized as a number between zero and one.[1] By deriving the first order condition with respect to consumption it is straightforward to show that

$$c_T \quad = \quad \frac{w}{1+K^{\frac{1}{\gamma}}}\,,$$

we can then write the analytical expression for the last period value function:

$$V_T(w) \quad = \quad \frac{\left(\frac{w}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma} + K\,\frac{\left(\frac{wK^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma}\,.$$

Then the problem that agents solve in the next to last period of life is:

$$V_{T-1}(w) \quad = \quad \max_{0\leq c\leq w} \frac{c^{1-\gamma}}{1-\gamma} + \beta E\, V_T(w-c)\,.$$

Using the previous results we can write

$$V_{T-1}(w) \quad = \quad \max_{0\leq c\leq w} \frac{c^{1-\gamma}}{1-\gamma} + \beta E \left[\frac{\left(\frac{\tilde{r}(w-c)}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma} + K\left[\frac{\left(\frac{\tilde{r}(w-c)K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma}\right]\right]\,.$$

Here in order to derive the first order condition with respect to consumption we assume, as in Lavhari and Srinivasan (1969), that the value function is differentiable and that the differential and expected value operators can be interchanged. The $f.o.c$ is then,

$$c^{-\gamma} - \beta E\,(\tilde{r}^{1-\gamma}) \left[\left(\frac{(w-c)}{1+K^{\frac{1}{\gamma}}}\right)^{-\gamma} \frac{1}{1+K^{\frac{1}{\gamma}}} + K\left[\left(\frac{(w-c)K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right)^{-\gamma} \frac{K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right]\right] \quad = \quad 0\,.$$

Then some algebraic manipulation allows us to write the $f.o.c$ as

$$c^{-\gamma} \quad = \quad \beta E\,(\tilde{r}^{1-\gamma}) \left(\frac{(w-c)}{1+K^{\frac{1}{\gamma}}}\right)^{-\gamma}\,.$$

[1] We also follow in this case the "egoistic" model of bequests.

Some more tedious algebra leads to the following expression for the decision rule in the next to last period

$$c_{T-1} = \frac{w}{1+\beta^{\frac{1}{\gamma}}\left[E\left(\tilde{r}^{1-\gamma}\right)\right]^{\frac{1}{\gamma}}\left[1+K^{\frac{1}{\gamma}}\right]},$$

that can be rewritten as

$$c_{T-1} = \frac{w}{1+\beta^{\frac{1}{\gamma}}\left[E\left(\tilde{r}^{1-\gamma}\right)\right]^{\frac{1}{\gamma}}+\beta^{\frac{1}{\gamma}}\left[E\left(\tilde{r}^{1-\gamma}\right)\right]^{\frac{1}{\gamma}}K^{\frac{1}{\gamma}}}.$$

Assuming next that the interest rate, $\tilde{r}$, follows a log-normal distribution with mean μ and variance σ^2, then given that $E(\tilde{r}) = e^{\mu+\frac{\sigma^2}{2}}$ and denoting $E(\tilde{r})$ as $\overline{r}$ we can write

$$E\left(\tilde{r}^{1-\gamma}\right) = \overline{r}^{1-\gamma}e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}}.$$

We then substitute back in the formula for c_{T-1} and obtain

$$c_{T-1} = \frac{w}{1+\beta^{\frac{1}{\gamma}}\left(\overline{r}^{1-\gamma}e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}+\beta^{\frac{1}{\gamma}}K^{\frac{1}{\gamma}}\left(\overline{r}^{1-\gamma}e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}},$$

given the similarity with expression (8) in the text it is easy to see how backward induction would lead us to the decision rules for the rest of the periods, for example we can write c_{T-k} as

$$c_{T-k} = \frac{w}{1+\beta^{\frac{1}{\gamma}}\left(\overline{r}^{1-\gamma}e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}+\beta^{\frac{2}{\gamma}}\left(\overline{r}^{1-\gamma}e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}+\ldots+\beta^{\frac{k}{\gamma}}K^{\frac{1}{\gamma}}\left(\overline{r}^{1-\gamma}e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}}.$$

We can also see that if γ is equal to 1 we are back to the logarithmic utility case and the expression for c_{T-1} above is equivalent to (8), which is a special case of the expression above. It is also important to emphasize that this expression is the finite horizon counterpart to the one obtained in Levhari and Srinivasan (1969) once a bequest motive is introduced, and that their results regarding the effects of uncertainty (decreasing proportion of wealth consumed as the uncertainty grows if $\gamma > 1$) go through in this case.