

# The Annuity Puzzle Revisited<sup>†</sup>

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## Abstract

There is a pressing need for a better understanding of how access to various types of financial products can impact retirement behavior, especially if this access comes from a change in the incentive scheme through a reform of the current Social Security system. This is especially important if we are to provide useful policy recommendations regarding reform to the current social insurance system. In this paper I focus on the “annuity puzzle,” the question as to why the annuity market is so narrow. I present a model that endogenizes the annuity decision along with the consumption/saving and labor supply decisions. This research enhances our understanding of how annuities work in a life cycle model with more realistic characterizations of the choices and incentives that individuals face. My results show that the low rates of annuitization can be the product of optimal decision making by individuals in a life cycle model which endogenizes the labor/leisure decision and accounts for Social Security. The government should pay particular attention to the rules regarding withdrawal of benefits through annuities or lump-sums when introducing individual retirement accounts or other privatization schemes, given the interaction between retirement incentives and the attractiveness of annuities.

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# 1 Introduction

There is a pressing need for a better understanding of how access to various types of financial retirement products can impact retirement behavior, especially if this access comes from a change in the incentive scheme through reforms of the current Social Security system. A body of new research is tackling some of these issues, but more attention needs to be given to solving some of the puzzles that the access to some of these financial instruments already represents, even in the absence of a widespread endorsement from public institutions. This is especially important if we are to provide useful policy recommendations regarding reform to the current social insurance system.<sup>1</sup>

In order to contribute to the objectives described above, this paper presents a dynamic model of the joint labor/leisure, consumption/saving and annuity decisions over the life cycle. I introduce several models of the life cycle decision making of the individual, in increasing level of complexity and closeness to reality, in order to provide a framework of policy analysis for considering important policy experiments related to the reforms of Social Security. I introduce in this consumption/saving and labor/leisure framework the possibility of endogenously choose annuities under capital uncertainty and in the presence of Social Security and bequest motives. The model provides new insights regarding the “annuity puzzle” and the effects of social insurance on labor supply and wealth accumulation.

I begin the analysis with a simple model, which ignores, for almost all purposes, the individual’s labor supply decision. In this model, consumption and saving over the life of the individuals are analyzed in detail. Modigliani and Brumberg (1954, 1980), Friedman (1957), Beckmann (1959), Phelps (1962), and Ando and Modigliani (1963) represent seminal contributions to the analysis of this classic problem in economics.

Phelps presents an infinite horizon model of the consumption/saving decision under investment uncertainty (providing the framework for the model that I solve first), and he derives closed-form solutions for several models with varying assumptions regarding the utility function. Hakansson (1970) provides a refinement and extension of Phelps’ work, allowing for a choice among risky investment opportunities and the possibility to borrow and lend.<sup>2</sup>

I first present a finite horizon version of the simplest model and report closed-form solutions for the consumption decision rule. I then solve this model numerically with two objectives in mind: first, to validate

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<sup>1</sup> A number of the reforms in other countries have been done before some of the details of the system were fully understood and their implications for individual behavior studied. For instance, in Chile, the system is facing the challenge resulting from low participation rates, relatively high reliance on a minimum pension offered by the government, and potentially early retirement rates which make problematic to maintain even a small minimum pension level.

<sup>2</sup> Levhari and Srinivasan (1969) re-examine Phelps’ model and include a dynamic portfolio choice. Merton (1969) generalizes Phelps’s model to the continuous time case and also allows for a portfolio selection decision. Samuelson (1969) analyzes the lifetime portfolio selection problem in discrete time. Fama (1970), assuming “perfect markets,” shows that a two-period model provides most of the insights of the multi-period model of consumption decisions.

the techniques that will be used exclusively in the more realistic model which introduces labor supply, annuities, and Social Security; and second, to determine whether an accurate characterization of the finite horizon problem is as difficult to obtain as it is for the infinite horizon case. Rust (1999a) discusses the complications involved in attempting to replicate Phelps' (1962) solutions using numerical dynamic programming.<sup>3</sup> The unboundedness of the utility functions used complicates the numerical approach, and even when using the most sophisticated techniques under the assumption of logarithmic utility, the problem remains quite challenging.

The numerical approach for the finite horizon case is fairly well behaved. Even in the absence of the bequest motive, the numerical solution approximates the closed form solution quite well, using either the logarithmic utility function or the CRRA utility function. I show both analytically and numerically that the finite horizon solution of the consumption/saving problem with bequests converges to the infinite horizon model (without bequests). I also show simulated solution paths for consumption and wealth accumulation.

Modified versions of this benchmark model of the consumption/saving decision have been used extensively in the literature with different objectives. Hubbard and Judd (1987) provide a partial and general equilibrium discussion of the importance of social insurance in a model with uncertainty and borrowing constraints. Thurow (1969) invokes credit market restrictions to reconcile the prediction of the life cycle model with the empirical evidence, in particular with the fact that consumption tracks income quite closely in the data. Zeldes (1989a) and Deaton (1991) study the role of liquidity constraints using extensions of this model in a finite and infinite horizon framework, respectively. Beckmann (1959) presents a dynamic programming model that introduces income uncertainty (but with no labor decision), Sandmo (1970) explores the role of income and capital uncertainty in a two period consumption/saving model, and Miller (1974) presents the infinite horizon version of such a model concentrating on income uncertainty, finding that agents would always consume less when income is stochastic. Nagatani (1972) also introduces income uncertainty to justify the close relationship between consumption and income in the data, and Zeldes (1989b) solves a similar model using numerical techniques, since closed-form solutions are unavailable when using a constant relative risk aversion utility function.

Skinner (1988) explores the importance of precautionary savings in a model with risky income, approximating the optimal consumption path via Taylor expansions. Carroll (1997) presents a theory of *buffer-stock saving* where individuals maintain contingency funds to hedge against income uncertainty. Some empirical evidence presented by Carroll (1994) and Carroll and Samwick (1997) seems to support certain implications of this theory. Hubbard et al. (1994, 1995) analyze and solve with numerical techniques, a multi-period

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<sup>3</sup> See also Benítez-Silva, Hall, Hitsch, Pauletto, and Rust (2000) for a discussion of this issue and a comparison of numerical methods for solving a wide range of problems in economics.

model of the consumption decision with uncertain lifetimes, and stochastic wages and medical expenses. They emphasize the importance of precautionary savings and the role of social insurance. Attanasio and Weber (1995), Attanasio and Browning (1995), and Attanasio et al. (1997) highlight the importance of considering the effects of changes in demographics and labor supply behavior in a life cycle model if we are to match the empirical evidence. However, they still model labor supply as exogenous. More recently Gourinchas and Parker (1999) have estimated the consumption/saving model using simulation techniques, and Cagetti (1999) has focused on wealth accumulation. Dynan et al. (1999) explore saving behavior across income groups, Banks et al. (1998) analyze income and expenditure patterns around the time of retirement, Engen et al. (1999) study the adequacy of household saving in a model with uncertain lifetime and income uncertainty, Palumbo (1999) highlights the importance of taking into account uncertain medical expenses to explain the slow rates of dissaving among the elderly, and Cifuentes (1999) uses the consumption/saving model to discuss the effects of Pension reform.<sup>4</sup>

None of these models considers explicitly the labor supply decision of the individual, and thus, our work can be considered an attempt to complement and extend those models by considering labor decisions as indeed endogenous to the life cycle consumption/saving problem. However, this is not a completely novel consideration, Heckman (1974), Heckman and MaCurdy (1980), MaCurdy (1981, 1983), Bodie and Samuelson (1989), Bodie, Merton and Samuelson (1992), Low (1998, 1999), Flodén (1998), and French (2000) tackle this issue in theoretical and/or empirical contexts, but our models attempt to incorporate realism by considering additional sources of uncertainty, introducing annuities and Social Security, and providing a general framework which allows for policy experimentation.

More recently, an increasing number of papers have incorporated the labor decision in general equilibrium models of the economy in their analysis of the effects of Social Security reform. Huggett and Ventura (1998), Büttler (1998), İmrohoroğlu et al. (1994, 1999a, 1999b), and De Nardi et al. (1999) are just a few examples. However, since they do not focus on individual decision-making, and because the general equilibrium approach requires a number of strong assumptions to make the problem solvable, there are many aspects of the life cycle model still to be addressed.

At the heart of our work is the allowance of agents to make their labor/leisure decision along with their consumption/saving decision in a utility maximizing framework in finite horizon. Individuals can work full-time, part-time, or not at all at any point during their life, and they can consume continuously subject to a budget constraint (they can not borrow against future income). They can also accumulate wealth over their life at an uncertain rate of return which I model as draws from a log-normal distribution. Following

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<sup>4</sup> Browning and Lusardi (1996) present an comprehensive survey of the consumption/saving literature and focus on saving behavior. See also Deaton (1992) for an illuminating presentation of consumption models.

our piecemeal approach to solving these models, I first introduce wages as deterministic; that is, agents know their exact profile of wages from day one. This effectively maintains the value function as dependent only on wealth, making the model a fairly simple extension of the consumption/saving model. I consider an isoelastic and Cobb-Douglas utility function on consumption and leisure, and given the unavailability of closed-form solutions when the marginal utility is non-linear, I solve the problem numerically by backward induction using dynamic programming techniques. I will assume throughout most of the analysis that the constant relative risk aversion parameter is larger than one, effectively implying that consumption and leisure are substitutes.<sup>5</sup> I also have to parameterize the within-period valuation of consumption versus leisure, a parameter that has an important effect on the labor supply decisions, as will become clear from our discussion in the following sections. I show that this model already captures paths of consumption, labor, and wealth accumulation, consistent with the literature and empirical regularities.

Next I introduce labor income uncertainty, allowing for the wages to be stochastic. I start by characterizing the wage realizations as independent and identically distributed draws from a log-normal distribution, with a mean at each point in time that matches both the deterministic profile considered previously and a standard deviation consistent with research on the variability of income. This new source of uncertainty increases the computational burden of solving the model by a single order of magnitude, since now the value function also depends on the uncertain draws of wages. The numerical techniques used can still handle the problem, but computing time increases as the “curse of dimensionality” makes its appearance. I then allow for serial correlation in the wages following the empirical evidence on the topic. I solve models with different serial correlation factors and compare the results to those of the previous models. I then simulate the solutions with certain starting values of the state variables and average out the simulations to compute a path for consumption, labor, and wealth accumulation over the life cycle.

I then tackle the problem of introducing annuities and a Social Security system to this framework.<sup>6</sup> The strategy is to first introduce in the consumption/saving model with bequests the possibility of partial or total annuitization by individuals and then extend the model and introduce endogenous labor and Social Security. Agents endogenously decide to annuitize at any point of their lives part or all of their wealth; that is, they can purchase a sure income stream for the remainder of their lives at a price which takes into account the average age specific mortality probabilities in the population.<sup>7</sup> The cost of the annuity cannot exceed current wealth in the period that they annuitize, and the decision to annuitize is unique and non-reversible. This last

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<sup>5</sup> See the discussion in Heckman (1974) and Low (1998).

<sup>6</sup> Rust (1999b) presents a survey of models that try to incorporate uncertainty and insurance mechanisms in models of social insurance.

<sup>7</sup> The literature refers to this type of annuity as *single premium immediate life annuity*.

assumption effectively means that they can only annuitize once in their life.<sup>8</sup> I do not, however, force them to do so at a given age or stage of their lives.<sup>9</sup>

To solve this model I have to take into account that agents are choosing the optimal time to annuitize and the size of the annuity along with their consumption/saving decision, forcing us to carry the annuity value as another state variable of the problem. This is an important exercise because I introduce this kind of insurance in the simplest possible stochastic model of lifetime decision making and show that agents do react to the availability of this insurance.

I then extend this model to consider the labor/leisure decision as endogenous and introduce Social Security. The full model provides important insights into the classic and important question of whether social schemes affect the behavior of individuals, and this model of endogenous annuities provides some insights into the “annuity puzzle,” the question as to why the annuity market is so narrow.

Our results suggest that the low rates of annuitization can be the product of optimal decision making by individuals in a life cycle model which endogenizes the labor/leisure decision and accounts for Social Security. This important result is consistent with the conclusions of Bodie and Samuelson (1989), and Bodie, Merton and Samuelson (1992). They all emphasize the role of labor supply flexibility in making risky investments more attractive.<sup>10</sup> In this paper I find that the counterpart of that result is that life annuities are a less attractive investment once the more complete model that endogenizes labor supply decisions is considered.

The policy implication of these findings are that the government should pay special attention to the rules for withdrawing balances from individual retirement account and other similar types of privatized financial instruments since the interactions between labor supply, retirement, and annuity decisions can lead to a large proportion of the population opting for lump-sum withdrawals. The model presented here can be used a first step in understanding some of the effects of a variety of rules that these privatized systems will have to take into account.

In the next section I solve analytically and numerically the finite horizon version of the consumption/saving benchmark model and simulate its implied consumption and wealth accumulation paths. Section 3 introduces the endogenous labor/leisure model, presents its numerical solution, and provides a discussion of its results. In section 4, I extend the life cycle models of consumption/saving and labor/leisure decisions to allow for endogenous annuitization and the presence of Social Security. Section 5 summarizes the main results and discusses extensions currently being considered and implemented.

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<sup>8</sup> This is a fairly realistic assumption, as emphasized in TIAA-CREF (1999).

<sup>9</sup> This model complements and extends the framework introduced in Friedman and Warshawsky (1990).

<sup>10</sup> In a recent paper Benítez-Silva (2003) has empirically tested this theoretical prediction and found that that individuals with labor supply flexibility hold up to 14% more wealth in stocks.

## 2 The Consumption/Saving Model

In this section I solve a finite horizon version of the consumption/saving problem analyzed in Phelps (1962).<sup>11</sup> Agents choose consumption according to the following utility maximizing framework:

$$\max_{0 \leq c_s \leq w} E_t \left[ \sum_{s=t}^T \beta^{s-t} u(c_s) \right], \quad (1)$$

where  $\beta$  is the discount factor, which for simplification in the derivations includes the mortality probabilities (later on these mortality probabilities will be considered separately in the solution and simulation of the model),  $c$  represents consumption, and  $w$  is wealth at the beginning of the period. Savings accumulate at an uncertain interest rate of return  $\tilde{r}$  such that  $w_{t+1} = \tilde{r}(w_t - c_t)$ . Utility depends only on consumption.<sup>12</sup> I can express and solve this problem using Dynamic Programming and Bellman's principle of optimality. I solve it by backward induction starting in the last period of life, in which the individual solves

$$V_T(w) = \max_{0 \leq c \leq w} \log(c) + K \log(w - c), \quad (2)$$

assuming a logarithmic utility function where  $K \in (0, 1)$  is the bequest factor.<sup>13</sup> By deriving the first order condition with respect to consumption I find that

$$c_T = \frac{w}{1 + K}, \quad (3)$$

and from this I can write the analytical expression for the last period value function:

$$V_T(w) = \log\left(\frac{w}{1 + K}\right) + K \log\left(\frac{wK}{1 + K}\right). \quad (4)$$

I can then iterate by backward induction and write the next to last period value function as:

$$V_{T-1}(w) = \max_{0 \leq c \leq w} \log(c) + \beta E V_T(w - c), \quad (5)$$

where the second term in the right hand side can be written as

$$E V_T(w - c) = \int_0^{\tilde{r}} V_T(\tilde{r}(w - c)) f(\tilde{r}) d\tilde{r}, \quad (6)$$

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<sup>11</sup> Phelps solved the infinite horizon problem analytically assuming no labor income and using different forms of the utility function.

<sup>12</sup> This is the standard characterization of the utility function. In a slightly different setup, Alessie and Lusardi (1997b) introduce habit formation, by considering a utility function that depends additionally on past consumption. See also Deaton (1992) for a discussion of such a model.

<sup>13</sup> Agents in this model care only about the absolute size of their bequests, leading to its been called the "egoistic" model of bequests. A bequest factor of one would correspond to valuing bequest in the utility function as much as current consumption. The importance of bequest motives is still an open issue in the literature. Here I take the position of acknowledging that bequests do exist and explore the implications of changing the importance of the bequest motive in the utility function. Hurd (1987, 1989), Bernheim (1991), Modigliani (1988), Wilhem (1996) and Laitner and Juster (1996) are some of the main references on the debate over the significance of bequests and altruism in the life cycle model. Kotlikoff and Summers (1981) stress the importance of intergenerational transfers in aggregate capital accumulation.

where  $\tilde{r}$  is the stochastic return on capital accumulation. Then I can write

$$V_{T-1}(w) = \max_{0 \leq c \leq w} \log(c) + \beta E \log(\tilde{r} \left( \frac{w-c}{1+K} \right)) + \beta K E \log(\tilde{r} \left( \frac{(w-c)K}{1+K} \right)). \quad (7)$$

Here the logarithmic utility simplifies the problem. Again taking first order conditions with respect to consumption, I obtain an expression for the consumption rule in the next to last period of life:

$$c_{T-1} = \frac{w}{1 + \beta + \beta K}. \quad (8)$$

I then have an expression for  $V_{T-1}$  in the following form:

$$V_{T-1}(w) = \log\left(\frac{w}{1 + \beta + \beta K}\right) + \beta \log\left(\frac{w\beta}{1 + \beta + \beta K}\right) + \beta K \log\left(\frac{w\beta K}{1 + \beta + \beta K}\right) + Y, \quad (9)$$

where  $Y$  gathers all the terms that do not depend on  $w$ . From here we can write  $V_{T-2}$  and again derive first order conditions, resulting in

$$c_{T-2} = \frac{w}{1 + \beta + \beta^2 + \beta^2 K}. \quad (10)$$

Through backward induction, I continue iterating to find  $c_{T-k}$

$$c_{T-k} = \frac{w}{1 + \beta + \beta^2 + \beta^3 + \dots + \beta^k + \beta^k K}. \quad (11)$$

for any  $k < T$ . From these decision rules, I can observe that as  $T$  grows large, the finite horizon solution with bequests converges to the infinite horizon solution, since the influence of the bequest parameter becomes less important as the time horizon increases. In the infinite horizon case with logarithmic utility and no non-labor income, the simplified decision rule is  $c = (1 - \beta)w$ , as shown in Phelps (1962). The derivation of the decision rules in the case of the CRRA utility function is similar though somewhat more involved and is presented in the Appendix. Benítez-Silva, Hall, Hitsch, Pauletto, and Rust (2000) present, among others, the CARA utility case under certainty.

The ability to derive an analytical solution for this model allows me to evaluate the effectiveness of the numerical methods, which are all that I have available in more complicated models. The exercise of solving the model numerically is also interesting on its own given that the infinite horizon version of this model has been shown to be quite difficult to replicate using numerical methods, even with the logarithmic utility function, as discussed in Rust (1999a) and Benítez-Silva, Hall, Hitsch, Pauletto, and Rust (2000).

The numerical procedure is by nature very similar to the analytical approach, involving backward recursion starting in the last period of life. I discretize wealth and compute the optimal value of consumption for all those wealth levels using bisection. Bisection is an iterative algorithm with all the components of a nonlinear equation solver. It makes a guess, computes the iterative value, checks if the value is an acceptable solution, and if not, iterates again. The stopping rule depends on the desired precision given that the

solution is bracketed by the nature of the algorithm and that the round-off errors will probably not allow us to increase the precision beyond a certain limit. In each iteration of the numerical solution, except for the final one where all uncertainties have been resolved, I have to compute the expectation in equation (6), which is potentially the most computationally demanding step. For this I use Gauss-Legendre quadrature. I also compute the derivative of this expectation using numerical differentiation, also requiring quadrature as part of its routine. Here the analytical derivatives are simple to compute, but this is not always the case for more complicated models. I therefore wish to evaluate the accuracy of the numerical strategy.

Gaussian quadrature approximates the integral through sums using rules to choose points and weights based on the properties of orthogonal polynomials corresponding to the density function of the variable over which I am integrating, in this case the draws of the interest rates following a log-normal distribution. The points and weights are selected in such a way that finite-order polynomials can be integrated exactly using quadrature formulae. The weights used have the natural interpretation of probabilities associated with intervals around the quadrature points.<sup>14</sup> At this point I am considering a one dimensional problem, for which quadrature methods have been shown to be very accurate compared with other techniques of computing expectations (integrals) such as Monte Carlo integration, Low Discrepancy sequences and weighted sums.<sup>15</sup>

This all amounts to manipulating (6) through a change of variables such that I can write it as an integral in the  $(0, 1)$  interval and then approximate it by a series of sums depending on the quadrature weights and quadrature abscissae which I compute recursively, following readily available routines (e.g. Press et al. 1992).<sup>16</sup>

An additional numerical technique that we use to solve the model completely is function approximation by interpolation. Since savings in a given period are accumulated at a stochastic interest rate, next period's wealth will not necessarily fall in one of the grid points for which I have the value of the function already calculated. Ideally I would solve the next period's problem for any wealth level, but this is computationally infeasible. Therefore, I use linear interpolation to find the corresponding value of the function given the values in the nearest grid points.<sup>17</sup>

The bisection algorithm that uses the quadrature and interpolation procedures eventually converges to a maximum of the lifetime consumption problem for a given value of wealth in a given period (or reaches

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<sup>14</sup> For a detailed characterization of quadrature methods I refer the reader to Tauchen and Hussey (1991), Rust (1996), Judd (1998), and Burnside (1999).

<sup>15</sup> For an analysis of how different techniques perform in applied problems see Rust (1997).

<sup>16</sup> I can write  $\int_r V(r) f r dr$  after a change of variables as  $\int_0^1 V(F^{-1}) du$ , which can then be approximated by  $\sum_{i=1}^N w_i V(F^{-1}(u_i))$ , where  $w_i$  are the quadrature weights and  $u_i$  are the quadrature abscissae.

<sup>17</sup> More sophisticated interpolation procedures can be used such as splines or Chebyshev interpolation. They are not considered here, Benítez-Silva, Hall, Hitsch, Pauletto, and Rust (2000) provide with a sensitivity analysis of the procedures used at each step of similar numerical computations.

the pre-decided tolerance level). This procedure is repeated until the solution of the first-period problem is obtained.

Once I have solved the model, I have a decision rule for every level of wealth in our initial grid. Here I have chosen a grid space of 500 points; to gain accuracy more of these points are concentrated at low wealth levels where the function is changing rapidly. Figures 1 and 2 show the decision rule of the consumption/saving problem for wealth ranging from 0 to 100 units. For expositional purposes I have solved a 10-period model.

Figure 1 plots several decision rules given logarithmic utility. It first plots the numerical solutions for different time periods, denoted  $C_1$ ,  $C_2$ , and so on. It also plots the solution of the infinite horizon problem borrowing from Phelps (1962), denoted by  $CINF$  in the figure. I have chosen a discount factor of 0.95 and a bequest parameter of 0.6. Figure 2 plots the decision rule when I consider a CRRA, with risk aversion parameter equal to 1.5,  $\beta = 0.95$ , and bequest parameter equal to 0.6, I also plot the analytical solution of the infinite horizon problem, borrowing from Levhari and Srinivasan (1969). For both types of utility function I observe that the consumption rules increase with wealth and time and that in very few periods I am fairly close to the solution of the infinite problems.

Figure 3 and 4 are concerned with comparing the numerical solutions with the true analytical solutions derived above and in the Appendix. I plot in both figures the percentage difference between the two solutions in terms of the value of the true solution, for a sample of time periods. The numerical technique performs quite well. For about half of the range of values, the numerical solution is very accurate with deviations below 1%, for both types of utility functions. After that, errors are a bit larger, especially for early time periods. For the first period and for high levels of wealth the error reaches 12% to 13%, depending on the utility function. These differences are the result of the extrapolation methodology for accounting for wealth levels outside the grid of points I am solving over. I extrapolate linearly, what in some cases can lead to a better than average return for the individual, this leads our agents to underconsume in order to profit from this advantage.

In Figures 5 and 6 I simulate this model using the numerical solution for the CRRA utility function. I now model separately the mortality probabilities at every age, following the U.S. Life Tables for 1997. That means that I assume all individuals die at age 85, and before that they are aware of the exogenous probability of dying at every age. I report the results of 5,000 simulations of an 61-period model (simulating an individual that starts making decisions at age 25 and dies at age 85) with 500 grid points for wealth in

the 0 to 200,000 range.<sup>18</sup> I plot consumption and wealth paths with an initial wealth level of 10,000.<sup>19</sup> I also consider several values for the parameters of interest. In the first specification,  $\gamma$  is taken to be 1.5 (the parameter of relative risk aversion), and it is increased to 2.5 in the second specification (*hg* lines in the plots). I then increase the bequest parameter to 0.6, leaving  $\gamma = 1.5$  (*bq* lines in the plots), and finally, I decrease the relative risk aversion parameter to 0.7 (*lg* lines in the figures).

I observe that people consume less at the beginning of their lives, with increased consumption in the final periods of life, given uncertain interest rates represented by draws from a truncated log-normal distribution. Consumption does, however, decrease if the risk aversion parameter is less than 1. Focusing on the pattern of wealth accumulation, we observe that individuals deaccumulate their wealth gradually. We also see that increasing the relative risk aversion parameter has the effect of making consumption less smooth (with higher wealth accumulation), while decreasing the parameter from the benchmark value of 1.5 leads to more smoothing (with lower wealth accumulation). We can also observe the expected effect of the bequest parameter: those with a higher concern for their offspring, represented by a higher valuation of bequests in the utility function, consume at almost every age less than do those with a lower bequest parameter. This former population also accumulates more when young. These results regarding the effect of the bequest motives are consistent with, and in fact extend, the theoretical model of Hurd (1987) to the case of agents with various levels of bequest.

This model is meant to serve as a benchmark for the models discussed next and for the introduction of annuities and Social Security in Section 4.

### 3 Introducing the Labor/Leisure Decision

I next tackle the issue of extending the model of Section 2 to allow for an endogenous labor supply decision. Utility is now a function of consumption and leisure, and agents will optimally choose both in every period of their lives. They effectively solve

$$\max_{c_s, l_s} E_t \left[ \sum_{s=t}^T \beta^{s-t} u(c_s, l_s) \right], \quad (12)$$

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<sup>18</sup> The simulations in this section and throughout the paper represent averages of the thousand of simulations performed. A simulation starts from the first period of working lives and through interpolation finds the optimal paths of consumption/saving (for the model in this section), leisure, and annuities after drawing from the distribution of the unobservables. I need starting values for wealth, wages, and annuities (if applicable), and using the same parameters as for the previously solved model we stored the paths of the relevant variables and then average them out.

<sup>19</sup> This is approximately the net worth reported by Poterba (1998), using the Survey of Consumer Finances, for individuals at the beginning of their working lives.

again in finite horizon. The within-period utility function is assumed to be Isoelastic and Cobb-Douglas between consumption and leisure in time  $t$ :

$$u(c_t, l_t) = \frac{(c_t^\eta l_t^{1-\eta})^{1-\gamma}}{1-\gamma}, \quad (13)$$

where  $\gamma$  is the coefficient of *relative risk aversion* and  $\eta$  is the valuation of consumption versus leisure.<sup>20</sup> Consumption and leisure are substitutes or complements depending on the value of  $\gamma$  as discussed in Heckman (1974) and Low (1998), with the cutoff approximately equal to 1.<sup>21</sup> In most of the analysis I will assume values of  $\gamma$  larger than 1, implicitly assuming substitutability between consumption and leisure. I will assume that the agent has only three choices with respect to the labor decision: part-time, full-time, or out of the labor force.<sup>22</sup> It is also important to emphasize that for computational convenience we have chosen a lower bound on leisure equal to 20% of the available time during a given period.<sup>23</sup>

### 3.1 Deterministic Wages

First, I will assume that wages follow a deterministic path which peaks around age 50 and then smoothly decreases. Given that I allow for consumption and leisure to influence each other using a CRRA utility function, and considering that I am concerned with corner solutions for the labor decision, the model can only be solved numerically. To do so I employ the techniques presented in Section 2.

I use Dynamic Programming to characterize this problem and again solve by backward induction. The individual in the last period now solves

$$V_T(w) = \max_{(0 \leq c \leq w + \omega(1-l), l)} U(c, l) + K U(w + \omega(1-l) - c), \quad (14)$$

where  $\omega$  represent wages and leisure (labor) is chosen among the three possible states. Once I obtain the optimal decision rules using the bisection algorithm, I then solve recursively. I can write the value function in the next to last period as

$$V_{T-1}(w) = \max_{(0 \leq c \leq w + \omega(1-l), l)} U(c, l) + \beta E V_T(w + \omega(1-l) - c). \quad (15)$$

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<sup>20</sup> See Browning and Meghir (1991) for evidence on non-separability of consumption and leisure within periods.

<sup>21</sup> Heckman presents a model of perfect foresight and shows that by introducing the labor supply decision it is possible to reconcile the empirical evidence on consumption paths with the life cycle framework, without resorting to credit market restrictions or uncertainty. Low's (1998, 1999) work is fairly close in nature to our analysis, although he abstracts from capital uncertainty but allows borrowing. French's (2000) model is also close to our work, although it focuses on the retirement decision. Flodén (1998) uses a two-period model to illustrate the importance of considering labor supply as endogenous when analyzing consumption and saving under uncertainty.

<sup>22</sup> I solve in this case a 30-period model to reduce the computational burden of the solution process, but in Section 4 I present the results of a 61 period model.

<sup>23</sup> Different values of this parameter have essentially no effect on the solutions presented below.

The value function still remains unidimensional since there is no uncertainty about the wages. I solve this model again by bisection, computing the expectations by quadrature and interpolating the values of the next period's value functions.

Once I have solved the model, I simulate it given starting wealth values. The capital uncertainty is characterized by draws from a truncated log-normal distribution. Figures 7-9 present plots of the paths of consumption, labor supply, and wealth accumulation resulting from this 30-period model, which I map into an age profile for expositional purposes. I set initial wealth equal to 10,000 units and consider varying levels of the relative risk aversion parameter, bequest motive, and the valuation of consumption versus leisure in the utility function.

These results have several interesting features. First, as can be seen from Figure 7, consumption tracks income for a significant amount of time before age 40, at which point the consumption path begins to flatten, finally decreasing by the end of the life cycle. I can also see that those who value leisure more ( $\eta = 0.5$  versus  $\eta = 0.7$ , *eta* in the figure) receive lower wages because they work mostly part-time, although they are able to maintain an average consumption level higher than their part-time wage level starting at about age 40, since some individuals choose to work full-time. The pattern of labor supply is equally interesting. Agents with a high valuation of consumption seem to work full time most of their lives, except at the beginning when their wages are low and they have initial wealth to smooth consumption. Later in life, our model is able to pick up the decrease in labor supply due to lower wages. It is important to emphasize that those with higher bequest motives (*bq* in the figures, bequest parameter equal to 0.6 versus 0.1 for the other curves) work more on average than those with lower bequest parameters. In Figure 9 I show the wealth accumulation over the life cycle implied by the model. The pattern here is fairly close to the estimated, simulated, and reported results of several papers (e.g., Hubbard et al. 1994, Attanasio and Weber 1995, Attanasio et al. 1997, Alessie and Lusardi 1997a, Alessie et al. 1997, Poterba 1998, and Cagetti 1999) reflecting empirical data quite closely. We see little accumulation early in life, and then after age 40 agents begin to accumulate higher levels of wealth which only decreases near the end of life. We can also see from the graph that those with higher bequest motives start deaccumulating their wealth later in life and than those with higher valuation for leisure start to accumulate earlier in life. Finally, those that are more risk averse start accumulating later in life and end up accumulating less resources than the rest of individuals. This model is broadly consistent with some features of the data that show very low savings rates among young individuals, with an increase only later in life.<sup>24</sup>

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<sup>24</sup> I have also simulated a model with initial wealth equal to 50,000 units. In this case the model predicts very similar behavior, except at the beginning of life when wealthy individuals delay their entrance into the labor force and consume out of their initial endowment.

### 3.2 Stochastic and Serially Correlated Wages

I next make the model more realistic by introducing income uncertainty, while maintaining the endogeneity of the labor/leisure decision.<sup>25</sup> I start by introducing stochastic *i.i.d.* wages from a log-normal distribution with a changing mean that follows the deterministic profile used above.<sup>26</sup> This feature complicates the model because the value functions now depend on the uncertain wage realizations. I write the problem solved by the agents in the last period of life as

$$V_T(w, \omega) = \max_{(0 \leq c \leq w + \omega(1-l), l)} [U(c, l) + K U(w + \omega(1-l) - c)], \quad (16)$$

where labor is again chosen among the three possible states. Once I obtain the decision rules numerically I can write the value function in the next to last period:

$$V_{T-1}(w, \omega) = \max_{(0 \leq c \leq w + \omega(1-l), l)} [U(c, l) + \beta E V_T(w + \omega(1-l) - c, \omega)]. \quad (17)$$

The functions for the earlier periods are again obtained recursively. The expectation  $E V_t(\omega(1-l) + w - c, \omega)$  appearing in the value functions for the different periods can be written as follows:

$$\int_0^{\bar{r}} \int_0^{\bar{\omega}} V(\tilde{r}(w + \tilde{\omega}(1-l) - c), \tilde{\omega}) f(\tilde{\omega}) d\tilde{\omega} f(\tilde{r}) d\tilde{r}. \quad (18)$$

The interpolation of the values of the next period value function has to be carried out in two dimensions, a slightly more cumbersome and slower procedure. The double integrals are again solved by Gauss-Legendre quadrature, but I use iterated integration since I am assuming independence of wages and interest rates.<sup>27</sup>

Figures 10-12 show the consumption, labor, and wealth accumulation paths for this model. The main difference from the case of deterministic wages is that individuals start to save and accumulate later in life, and work on average a bit more later in life, ultimately accumulating a higher level of wealth before they enter the deaccumulation phase.<sup>28</sup>

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<sup>25</sup> I do not allow here for nonzero correlation between income shocks and asset returns. For a discussion of this possibility at the micro level see Davis and Willen (2000).

<sup>26</sup> An important parameter to be chosen here is the variance of the stochastic component of wages. I use in this case values of this parameter from estimations of the variance of innovations to wages. I use a number that is equivalent to say that one standard deviation of the innovation in wages accounts for around 10% of wages. See French (2000) for a recent contribution to this literature and the references therein.

<sup>27</sup> Given that the value function depends on wealth and wages, we needed to discretize both variables in order to approximate the integrals, using 50 points for wealth and 50 points for wages. I found that using fewer points significantly affected the accuracy of the calculations, leading to possible erroneous conclusions.

<sup>28</sup> Lusardi (1998) presents empirical results pertaining to the role of income variance in a consumption/saving model. She finds that income variation seems to affect precautionary savings, but the final effect on wealth accumulation is not too large. Our results indicate that individuals are using their labor supply to hedge the income uncertainty, meaning that they increase or decrease leisure depending on the draw of wages they face, resulting in a smaller effect of uncertainty on the other variables. That is why going from deterministic wages to stochastic one does not have a very large impact in this model. Low (1998, 1999) also makes this point.

Finally, I introduce serially correlated wages, such that

$$\ln \omega_t = (1 - \rho) \alpha_t + \rho \ln \omega_{t-1} + \varepsilon_t, \quad (19)$$

where  $\alpha_t$  is a quadratic trend that mimics the one presented in the case of deterministic wages. The  $\varepsilon_t$  are *i.i.d.* draws from a normal distribution with mean 0 and variance  $\sigma_\varepsilon^2$ . If  $\rho$  is 0, this reduces to the case of *i.i.d.* wages. The solution method does not change significantly from the last model, and only the careful manipulation of the serially correlated component has to be considered.

Figures 13-15 show the paths of the relevant variables. Our results do not show striking differences with the previous graphs. Consumption profiles again track income paths very closely up to age 45, when wealth accumulation starts in meaningful amounts. Higher serial correlation leads to accumulation and deaccumulation slightly later in life, since individuals seem to take advantage of the effects of serial correlation once their peak earnings years have been reached. The labor supply profile is apparently quite similar to those shown before, with individuals facing higher serial correlation in their wages working a bit longer than the rest. However, the reader should notice the trend in the simulated labor supply from figure 8, to 11, to 14. As uncertainty grows individuals reduce their participation later and later in life because they are indeed using their labor supply decision (that is now endogenous) to hedge that additional uncertainty. Therefore, in Figure 14, although wages are declining (the price of leisure is going down), individuals stayed in the labor force because of the uncertain trajectory of wages ahead of them. I plot the paths for different values of the serial correlation parameter. With high correlation, we plot the case of individuals starting with wealth of 10,000 units and initial wages of 30,000 units, the initial wage for those with low serial correlation is 20,000 units.

From the solution and simulation of these models I can conclude that a life cycle model with endogenized labor supply behaves quite consistently with the empirical data on wealth accumulation and consumption profiles and that wealth accumulation seems to start only in mid-life. Additionally, such a model captures the gradual exiting from the labor force by older individuals who face lower wages, declining uncertainty, and who have a lower serial correlation of wages once they reach a certain age. This model seems well-suited for analyzing important policy issues regarding the effects on savings and labor supply of reforms in social insurance programs.

## 4 Endogenously chosen Annuities

In this section I extend the models presented in Section 2 and Section 3.1, the consumption/saving model and the extended model of endogenous labor with deterministic wages, by allowing individuals to purchase

an annuity with a fraction or all of their wealth at any point in their lives. I also introduce a stylized Social Security system in the endogenous labor/leisure model with annuities. I endogenize the annuitization decision by providing the agents with the possibility of exchanging a certain number of dollars today for a stream of income over the rest of their lives. The annuity has a given rate of return computed using the average age-specific mortality probabilities in the population, making it an actuarially fair annuity. In the simulation I can easily make the annuity less than actuarially fair and analyze the effects on individual behavior. The cost of the annuity, calculated as the net present value of the promised stream of income, cannot exceed the total wealth of the agent at the time of the purchase of the annuity. This is a *single premium immediate life annuity*. The decision to annuitize is unique and non-reversible. These last two assumptions mean that individuals can only annuitize once in their lives. I do not, however, place any restriction on the timing of this annuity.<sup>29</sup>

The first model presented here is similar to that of Friedman and Warshawsky (1990), although they focus on older individuals and on the issue of annuity pricing in order to explain the almost non-existence of a market for these instruments. Another difference is that they force individuals to invest a proportion of their wealth in an actuarially fair social annuity, without considering investment uncertainty. Brugiavini (1993) focuses on the role of longevity uncertainty in the purchase of annuities in a two/three period model. She also considers a model that allows for income uncertainty and the different behavior of employees and entrepreneurs. Mitchell et al. (1999) use the term structure of interest rate rather than a fixed interest rate, to calculate the expected present discounted value of the annuities in a model of uncertain lifetime. They find that retirees should value annuities even if they are not actuarially fair. Brown (1999a), extending the model of Yaari (1965), focuses on the role of annuities when individuals face an uncertain lifetime, using data on older Americans to construct a measure of the consumer's valuation of additional annuitization.<sup>30</sup> However, his model abstracts from capital uncertainty and does not endogenize the annuity decision in the general sense that I do. Brown (1999b) uses data on older individuals to test and ultimately reject the "Annuity Offset Model," the hypothesis that old individuals purchase term insurance to offset the excessive annuitization imposed by the government social programs. Kotlikoff and Spivak (1981) also use a Yaari-type model to emphasize the important role of the family as an incomplete annuities market, with the annuity decision made at exogenous points in time. Eichenbaum and Peled (1987) use a two period model to underline the over accumulation of private capital in a model of competitive annuities with adverse selection.<sup>31</sup>

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<sup>29</sup> I do not consider at this point the role of taxes in the decision to annuitize, see Gentry and Milano (1998) for a discussion of the effects of taking taxes into account.

<sup>30</sup> Davies (1981) extends Yaari's model of uncertain lifetime and uses it to explain the low levels of deaccumulation by the elderly. Sheshinski (1999) also extending Yaari's theoretical model, accounts for retirement and consumption decisions, finding that continuous annuitization is better than annuitization at retirement.

<sup>31</sup> Walliser (1997, 1998) discusses the role of annuities in a social insurance framework, Boskin et al. (1998) provide an overview

The most important differences between our analysis below and that of previous research is the consideration of the labor/leisure decision and the introduction of a fairly realistic social security system, changes which yield striking effects on the results.

The agents are again choosing consumption in order to maximize utility over their lifetime but now have the choice of converting part of their wealth to an annuity. This annuity is actuarially fair in the sense that its rate of return depends on the average mortality probabilities in the population, and it provides a stream of income until the time of death, which can be considered uncertain given that I introduce mortality probabilities.<sup>32</sup> The annuity premium  $A(a)$ , where  $a$  is the annuity received every period and  $s_t$  are the survival probabilities for every age, is equal to

$$A = a \left[ \sum_{v=t}^T \beta^{v-t} \prod_{j=1}^{v-t} s_{t-1+j} \right] \quad (20)$$

where I define  $\prod_{j=1}^0 = 1$ , and I am assuming that agents receive the first payment in the same period in which they annuitize. I again solve this model by backward induction using numerical Dynamic Programming techniques.

#### 4.1 Endogenous Annuities in the Consumption/Saving Model

I first analyze the introduction of endogenous annuities in the consumption/saving model without labor. The decision in the last period of life is very similar to that of the consumption/saving problem, but now the value function depends not only on wealth but also on the value of the annuity, which enters the budget constraint:

$$V_T(w, a) = \max_{0 \leq c \leq w+a} U(c) + K U(w - c) . \quad (21)$$

In this last period I do not allow for the annuity decision to occur, since annuitizing would return exactly what they put into the annuity, assuming no transaction costs. But even if agents do not actually decide to annuitize, it is possible that they have annuitized earlier in their lives; thus, I must solve for the value function under as many combinations of wealth and annuity values as possible.<sup>33</sup> In the simulation part of the model, agents reaching the last period of life without having annuitized will not annuitize in the last period. Recall that agents still face capital uncertainty and life time uncertainty.

I can then write the next to last period value function as follows:

$$V_{T-1}(w, a) = \max_{0 \leq c \leq w-A(a)+a} U(c) + \beta E V_T(w, a) , \quad (22)$$

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of the role of annuities in the economy, and Brown et al. (2001) analyze the financial aspects of annuity markets.

<sup>32</sup> The concept of “actuarially fair” falls slightly short to define the financial instrument I am allowing agents to purchase, because they are also a riskless asset, as opposed to the risky alternative capital investment.

<sup>33</sup> As in the previous section, I discretize the two variables that enter the value function in order to approximate the integrals and again choose 50 grid points for each variable.

where  $A(a) \leq w$ , and to simplify the derivation I have assumed a constant mortality rate, again this will not be assumed in the formal solution and simulation of the model. Agents who have already annuitized will receive a stream  $a$  and subsequently find the optimal consumption rule. Agents who have not already annuitized are able to decide what portion of their wealth will be put into the annuity.

In order to solve this model I conduct a maximization in stages. First, for a given value of the annuity I compute the optimal consumption rule via bisection, and again use quadrature and interpolation to calculate the expectations (the integrals in the model). This is embedded in another bisection algorithm for calculating the optimal fraction of wealth to annuitize and the implied annuity to be received in the periods ahead, possibly 0. This can be written as

$$\max_a \max_{0 \leq c \leq w - A(a)} [U(c) + \beta E V(a, \tilde{r}(w - A(a) - c))], \quad (23)$$

and again  $A(a) \leq w$ . The first order condition for an optimum in the inner maximization is

$$U'(c_a) - \beta E [r V'(a, \tilde{r}(w - A(a) - c_a))] = 0. \quad (24)$$

I solve this by the same methods explained above. The outer maximization solution method is very similar, but now the first order condition results from differentiating

$$U(c(a, w)) + \beta E [V(a, \tilde{r}(w - A(a) - c(a, w)))] \quad (25)$$

with respect to  $a$ . This results in the following first order condition:

$$U'(c(a, w)) \frac{\partial c}{\partial a} + \beta E \left[ \frac{\partial V}{\partial a} - \tilde{r} \frac{\partial V}{\partial \tilde{w}} [A'(a) + \frac{\partial c}{\partial a}] \right] = 0, \quad (26)$$

which by the envelope condition reduces to this intuitively plausible *f.o.c.*:

$$E \left[ \frac{\partial V_{t+1}(a, \tilde{w})}{\partial a} \right] = E \left[ \tilde{r} \frac{\partial V_{t+1}(a, \tilde{w})}{\partial \tilde{w}} A'(a) \right], \quad (27)$$

where  $\tilde{w}$  is wealth next period. The left hand side of this expression can be understood as the marginal value of an additional unit of annuity and the right hand side as its marginal cost. The agent will try to set these equal when calculating the optimal annuity in every period.

Bisection searches over the values of the annuity (which imply optimal levels of consumption calculated by the inner bisection algorithm), using quadrature to calculate the expectations and again interpolating the values of the next period value function. The interpolation has to be performed in two dimensions, which complicates and slows down the procedure slightly.

Once I have solved this model, I simulate it and construct consumption, wealth accumulation, and annuity paths over the life cycle. The results are quite striking. In Figure 16 I replicate the model of Section

2 for a starting wealth value of 10,000 units in a 61-period model, which I then map into a lifetime age profile. Consumption is again increasing over the lifetime due to the investment uncertainty as well as the mortality uncertainty that is now explicit. Figure 17 shows the consumption path resulting from averaging 2,500 simulations for individuals with a starting wealth value of 10,000 units. We can see the smoothness of the path compared with that of Figure 16, for the same starting value of wealth and the same parameter values. In fact, consumption is practically flat around 400 units. The contrast is sharper for increased values of the parameter of relative risk aversion as shown in Figures 16 and 17.

In Figure 17 I also show the average annuity value received ( $a$  in the graph), which changes slightly at the beginning but remains mostly flat over the course of the lifetime. I report two different specifications: the first has a relative risk aversion parameter of 1.5, and the second ( $hg$  in the plot) has  $\gamma = 2.5$ .<sup>34</sup>

A higher relative risk aversion leads to less smooth consumption and wealth accumulation is higher, as we see in Figure 18, something consistent with the idea that higher risk aversion should lead to less smooth paths for lifetime consumption and higher paths of accumulation. Once annuities are available more risk averse individuals are also eager to smooth their consumption, and purchase the annuities early in life.

Figure 19 reports wealth accumulation and the evolution of the value annuitized ( $A$  in the graph) at each stage of the life cycle. There is a clear difference between this wealth path and that of Figure 18, which replicates the model of Section 2, implying that once individuals are able to annuitize they prefer to spend their wealth buying the annuity very early in their lives, with more risk averse individuals again benefiting the most. The annuitization happens very early in this endowment consumption/saving model with investment uncertainty, and for average agents, amounts to more than half their wealth in the initial periods. Depending on the realizations of interest rates agents sometimes annuitize later in life and in a lower proportion, a seemingly reasonable result. These results are also consistent with Mitchell et al. (1999) suggesting that our model extends their simplified stochastic life cycle model to a full dynamic characterization of the annuitization decision in the presence of bequest motives and capital uncertainty, allowing for annuitization to happen at any point in the life cycle and with any fraction of the individual's wealth.

I have also simulated a model with less than actuarially fair annuities, by calculating the annuity receipts assuming that the insurance company multiplies the actual mortality probabilities by a factor  $\lambda < 1$ . Annuity is still chosen by individuals but now lifetime consumption and the annuity receipts are uniformly lower.

These results have several interesting implications. First, in a simple model of consumption and saving

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<sup>34</sup> Here I consider actuarially fair annuities. Non-actuarial fairness seems to be very common in this market, this can result from the fact that the insurance companies selling the annuities face transaction costs and adverse selection. The latter due to the fact that potentially healthier individuals are more likely to buy the annuity and therefore obtain an implicit higher rate of return from their purchase.

decisions with investment uncertainty, the possibility of annuitizing wealth is used by individuals to smooth their consumption stream almost entirely. If we interpret this mechanism as a pseudo-social insurance system, there is no doubt as to the importance of the effects that such a scheme has on the microeconomic behavior of agents. To make this point clearer Table 1, in its second column, provides calculations of the welfare effects of introducing annuities in this consumption/saving model. The table shows the equivalent variations in percentages of current wealth for individuals of different ages and initial level of resources. The equivalent variation as expressed here, provides a measure of the fraction of wealth a given individual is willing to give up to have access to the annuity market. We can see in the table that regardless of age and initial financial conditions the access to annuities is highly valued, individuals are willing to give up between 50% and 60% of their wealth to be able to purchase the annuity and smooth consumption as we saw in Figure 17. These results come to reinforce the conclusions of Mitchell et al. (1999) in terms of welfare effects annuities. I should also emphasize that I have done the same calculations but allowing for some degree of actuarial unfairness and the results are that even in that case the gains from having access to the annuity market is substantial and of a similar order of magnitude.

However, I want to explore the reasons for the lack of availability of such annuities in the current capital markets. Some researchers emphasize the issue of pricing, and some point to adverse selection; yet others blame it on the high capital returns to equities. Our average results seem to provide some insights into the “annuity puzzle,” the question of why the annuity market is so narrow. If it is optimal for an average individual to annuitize between 50% and 70% of their wealth, as our model suggests, and Social Security accounts for approximately that proportion of their wealth (Friedman and Warshawsky 1990), it is very likely that the low demand for annuities that we observe is the result of optimal decision making by individuals. For some individuals Social Security would provide less than the optimal level of annuitization, causing them to buy additional annuity notes in the market. For others, S.S. would lead to over-annuitization and they might react by buying life insurance to offset the imposed annuity purchases through the social insurance system.

Even if the reader agrees with this line of reasoning we are still left with an explanation that comes from outside the model I am solving, interestingly, the most novel results and insights come from the extension of this model to endogenize the labor decision and introduce Social Security, which I consider next.

## **4.2 Endogenous Annuities in the Extended Framework**

Our conjecture regarding the effects of extending the classical life cycle model with annuities to endogenize labor supply, in the same fashion as in Section 3, is twofold. First, such a model could help shed new light on long standing questions such as the effect of Social Security on the micro behavior of agents. Second,

it is likely to provide further insights into the “annuity puzzle.” The conjecture regarding annuities is that once we introduce labor supply we should see the annuity decision delayed in the life cycle, given that individuals use their labor as an insurance instrument when they are young. The results confirm some of these conjectures, and go even further.

I once again proceed by numerical dynamic programming to solve a model of endogenous consumption/saving, labor/leisure, and endogenous annuities, employing backward induction. I can write the individual’s problem in the last period of life as

$$V_T(w, a) = \max_{(0 \leq c \leq w + \omega(1-l)\tau + a + ss, l)} U(c, l) + K U(w + \omega(1-l)\tau - c + a + ss), \quad (28)$$

where  $\tau$  represents the Social Security tax I discuss below, and  $ss$  the Social Security benefits to which the individual is entitled. I then obtain the optimal decision rules using the sequential bisection algorithms discussed in the previous subsection and solve recursively. I can write the value function in the next to last period as

$$V_{T-1}(w, a) = \max_{(0 \leq c \leq w', l)} U(c, l) + (1 - s_t) \beta E V_T(w', a) + s_t K U(w'), \quad (29)$$

where  $w' = w + \omega(1-l)\tau - A(a) + a + ss$  and  $s_t$  are the age-specific mortality probabilities. The computational burden of this model is similar to the model without labor supply since I assume a deterministic path of wages. I solve this model again by bisection, computing the expectations by quadrature and interpolating the values of the next period’s value functions.

An additional extension, already considered in the formulation of (28) and (29), is to introduce Social Security, and I do so in a stylized manner. Assuming a deterministic path of wages, and further assuming that I am analyzing the behavior of an individual born in 1937, that will be 65 as of the year 2002, I can compute the benefits that such a person will receive had he worked at least 35 years since the age of 21.<sup>35</sup> I follow the formulae provided in SSA (2000) and assume that individuals can only start receiving benefits at age 65. I do, however, allow for work after that age and control for the earnings test provision to calculate the corresponding benefits.<sup>36</sup> I also tax wages at the current individual tax rate (6.2%), free from the Disability Insurance withholding (given that I do not model Disability Insurance in this framework), and add the taxes paid by the employer (6.2%), since those payments can be considered as discounted from a theoretical *before*

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<sup>35</sup> This is a fairly simplified characterization of the current Social Security system, which in our model could result in an artificial trend towards early retirement, given that agents could potentially foresee the gains from not working or working less as age 65 approaches, because their benefits will remain the same regardless of their working history. Our results seem not to suffer from this problem, as we will see below. Thus, although stylized, our characterization of the current Social Security system is a good first approximation to evaluate behaviorally meaningful responses.

<sup>36</sup> See Myers (1993) for a comprehensive review of Social Security rules, and Friedberg (2000) for a discussion of the effects of the earnings test on labor supply. The Social Security Administration website is an excellent source of information not only for recipients, and future recipients, but also for researchers: [www.ssa.gov](http://www.ssa.gov).

*all taxes salary.*<sup>37</sup>

The results from this model are presented in Figures 20-25. The figures present the paths of consumption, labor supply, annuities, and wealth chosen optimally by individuals over their life cycle. I can compare these results both with the ones presented in the previous subsection and with the ones in Section 3.1. The most important effects of introducing labor supply are twofold: first, the annuity decision is delayed from the initial periods of the life cycle (around the early 20's in the previous model) to around mid-life; second, the average individual now annuitizes a smaller proportion of his or her wealth, becoming a less important insurance instrument for these agents. This effect is even stronger once I introduce Social Security, with annuitization becoming even more marginal compared with the overall resources of individuals at any given age. This last result sheds new light on the “annuity puzzle,” leading us to conclude that in a more complete dynamic framework it is less of a puzzle why annuities are less attractive as an insurance instrument than has been believed.

This conclusion is even stronger once I calculate again the welfare effects of introducing an annuities market. Table 1 shows in columns three and four the equivalent variations in terms of percentage of wealth resulting from introducing annuities in a model with the labor/leisure as endogenous in the absence or presence of Social Security. The difference in welfare effects are striking specially for young individuals, and also those with higher initial resources. Once the labor/leisure decision is endogenized and utility depends not only on consumption but also on leisure the welfare gains from having access to the annuities market are much smaller, dropping from around 60% to single digits. I can conclude that in this extended model annuities are still valued by individuals but to a much lesser extent than in the simpler consumption/saving model that all previous work in the area has considered.

A very important issue to highlight at this point, which is also valid for the model presented in the previous subsection, is that this partial and residual annuitization by the average individual is consistent with the theory that says that an individual would in principle annuitize all its wealth if the annuity were actuarially fair. The behavior of a single individual in our model in front of the possibility of purchasing an actuarially fair annuity, given a state of the world and the expectations over the future states of the world, is more heterogenous than the average results (product of thousands of simulations) show. Some individuals never annuitize, while others annuitize very late in life, but basically all of those who annuitize, no matter at what age, put 100% of their current wealth in that annuity, as a portfolio selection approach to this problem would tend to predict. Figures 24 and 25 show the effects of considering a less than actuarially fair annuity in the model, annuitization is reduced further, delayed in time and in many cases no annuity is purchased

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<sup>37</sup> The Internal Rate of Return (IRR) of the Social Security system that I introduce is 1.6522%, a fairly realistic number given the assumptions made to compute the benefits. See the discussion in Geanakoplos et al. (1999).

over the life time.

In some respects the extended model presented in this section is close to that of Bodie and Samuelson (1989), and Bodie, Merton and Samuelson (1992), in that it puts together the life cycle consumption/saving model with the portfolio decision, allowing for flexible labor supply. Their research concentrates in the continuous time case, and the effect of making labor supply flexible in the investment mix by individuals, and not on the consumption/saving and labor/leisure choices over the life cycle. Furthermore, they do not consider annuitization in their model. However, I consider that the insights from this work complement their results quite nicely. They find that allowing for labor supply flexibility in the traditional life cycle consumption/saving model leads to more investment in the risky asset, assuming negative correlation between investment returns and the labor income innovations. I find that even in the absence of this correlation, introducing labor supply in the model, allow us to explain why individuals would not choose annuities as their preferred investment product.

The richness of the model allows me to go even further, and I can show two very important effects of Social Security on behavior. First, we can see the labor supply response of individuals to the reception of Social Security benefits: they decrease their participation on average to part-time work when they reach age 65, but in some individual cases, complete retirement is chosen.<sup>38</sup> Second, the effect on savings and individual capital accumulation are not too striking, reducing them slightly and leading to an increase in accumulation after age 65. These results complement and extend the classic discussions of Feldstein (1974), Kotlikoff (1979) and Hubbard (1987), regarding the effects of Social Security on saving behavior by allowing for the annuities market to play a role along with the public social insurance system.<sup>39</sup> In both cases these results give important insights into classic questions regarding the role of Social Security in shaping individual behavior.<sup>40</sup> Another interesting exercise that this model allows us to perform is to assess the labor supply effects of the recent elimination of the earnings test for individuals 65 and older. The simulations shown in Figure 21 suggest that the effects can be substantial, leading to a sizable increase in labor supply

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<sup>38</sup> As discussed by Rust and Phelan (1997), to show the effects of Social Security on labor supply is a surprisingly difficult task. Task that their work and our model achieve successfully. French (2000) also highlights the effects of social insurance on the labor supply of the current and future old. However, in most cases our model predicts only partial retirement. There are several ways of making this model match the data better by introducing realistic features in the model; first, it is not difficult to argue that the within period valuation of consumption vs. leisure in the model can be changing over the life cycle, and if the valuation of leisure increases with age, along with a S.S. effect, we could easily see a clearer trend towards full retirement, the chosen value of  $\eta$  that I chose is completely arbitrary since we do not find in the literature any reliable estimate for this parameter; second, it would also be realistic to introduce health status as an important variable in the life cycle model that could again have an effect on this valuation of consumption vs. leisure, leading people with different health conditions to change their valuations, what could potentially lead to a clearer retirement trend; finally a more realistic characterization of the Social Security system coupled with the introduction of health insurance considerations could have a stronger effect on the hazard rates after the early retirement or normal retirement age.

<sup>39</sup> Page (1998) provides a survey of the empirical literature which tackles this issue.

<sup>40</sup> Imrohoroglu et al. (1999b) using a different model find results qualitatively similar to those reported here regarding the decline of labor supply and wealth accumulation once Social Security is introduced.

among those 65 to 69.

Finally, I can compare the welfare of individuals with and without Social Security in the extended consumption/saving model with endogenous labor supply and endogenous annuities. I compute how much extra wealth we would have to give to individuals at the beginning of life to make them as well off in a world with Social Security as they would be in a world without public social insurance. Remember that these individuals have access to risky assets and fairly priced annuities. The results show that depending on the initial level of wealth and annuities the (negative) compensating variation can be substantial, suggesting that a young individual would be better off in a world without Social Security.<sup>41</sup> I refer the reader to Rust, Buchinsky, and Benítez-Silva (2002) for a more complete discussion of the welfare effects of Social Security, they present a more formal model of the Social Security rules and allow for income uncertainty and show that young individuals would have to be compensated due to the existence of Social Security but that those over the age of 40 seem to be willing to pay to keep the social insurance system in place.

## 5 Conclusions

This paper has presented several models of life cycle consumption/savings and labor/leisure decision making under uncertainty. I first present a benchmark finite horizon consumption/saving problem and solve it analytically and then use numerical dynamic programming techniques to validate the methodology used throughout the paper. I find that the decision rule of the finite horizon model with bequests converges to the infinite horizon solution. I also find that numerical methods approximate the finite horizon version of Phelps (1962) model quite well. I then present a model that endogenizes labor supply, allowing first for deterministic wages, and then introducing income uncertainty. I conclude that the model is consistent with consumption and wealth accumulation profiles in the data and that precautionary savings can even increase when I consider that labor supply (another source of accumulating precautionary balances) is endogenous, a result consistent with Low (1998). The model also shows the reduction of labor force participation at the end of the life cycle.

The paper then introduces the possibility of endogenously choosing annuities in a consumption/saving framework with capital uncertainty, life uncertainty and bequest, later extended to endogenize the labor/leisure decision. Agents can choose to annuitize part or all their wealth at any point of their lives,

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<sup>41</sup> These welfare effects of Social Security have to be taken with caution given that, first, I am ignoring general equilibrium effects that can be substantial specially when such a radical policy change is implemented; second this model does not include income uncertainty, and it is reasonable to believe that in the presence of income uncertainty Social Security has an additional intrinsic value for individuals; and third that I am ignoring possible risk pooling or risk sharing in the household that would make the introduction of a Social Security system have less of an impact on individual behavior, leading to lower overall welfare effects (see e.g. Kotlikoff and Spivak 1981, and Kotlikoff, Spivak and Shoven 1987, for a discussion of this last point).

but they can do this only once. This model can be understood as a privatized system with no mandatory contributions, but with a one-time opportunity to annuitize. I then include a more traditional Social Security system. The solution is consistent with some early results in the literature and in a sense generalizes those models. I find that in the simple consumption/saving model agents do choose to annuitize a large portion of their wealth and that they do so early in life, allowing them to smooth consumption considerably compared with the behavior observed in the benchmark model. I provide welfare comparisons that show how highly valued is for individuals the access to the annuities market. Once I take into account the labor decision, annuities are bought later in life and on average represent a small percentage of average wealth holdings, an effect which becomes even clearer when I introduce Social Security. The welfare comparisons for this case show that in the extended model annuities increase welfare only slightly. I also show that labor supply and wealth accumulation react to the incentives set forth by Social Security, and that a young individual would have to be compensated with a substantial increase in wealth to be as well off in a world with S.S. as in a world without it. I claim that this complete model of endogenous consumption, labor, and annuity decisions provides important insights into the “annuity puzzle” since the lack of demand for annuities can be the result of optimal behavior once labor supply and Social Security are accounted for.

The policy implications of these results indicate that the government should pay special attention to the rules affecting withdrawal of funds in any privatized scheme to be considered since the interaction of these rules with the retirement and labor supply decisions can lead to individuals avoiding annuitization when possible if incentives are not properly studied. The model presented here can be considered a tool to experiment with the possible rules governing a privatized system where annuitization will be part of the financial options that individuals are presented with.

There are several possible extensions of the model(s) presented here. Rust, Buchinsky, and Benítez-Silva (2002) extend this model and that of Rust and Phelan (1997) to account for disability insurance and Medicare, and model more closely retirement and social insurance incentives. I am also planning to allow for added uncertainty through health shocks which can be correlated with wages, as well as mortality uncertainty based on life tables, instead of embedding it in the discount factor. Another extension would explicitly consider borrowing, as in the consumption/saving literature. The model could eventually also allow for private pensions, which are in a sense proxied by the private annuities in the current model. Our model can also be used to estimate underlying parameter values following the simulation techniques in Gourinchas and Parker (1999), and French (2000) given data on the variables of interest.

Finally, another extension of this model attempts to integrate the job search decision into the life cycle dynamic maximization framework introduced here (See Benítez-Silva 2002). Both young and older workers search for new jobs while out of work and on the job in non-trivial proportions. This activity should be taken

into account in a life cycle model given the importance of the outcomes for the future path of earnings, wealth accumulation, and lifetime utility. Such a unifying framework would extend the life cycle utility maximization model and reconcile these two bodies of literature, which although theoretically intertwined (See Siven 1974, and Seater 1977), have evolved in different directions.

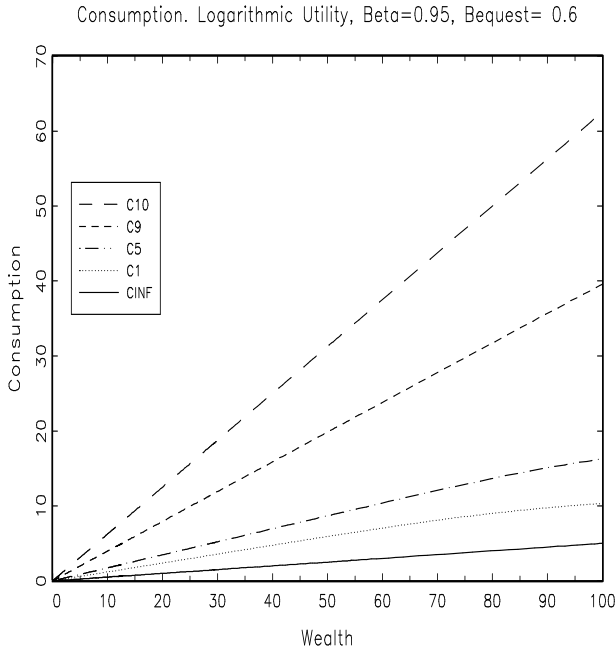


Figure 1: Consumption Decision Rule. Logarithmic Utility

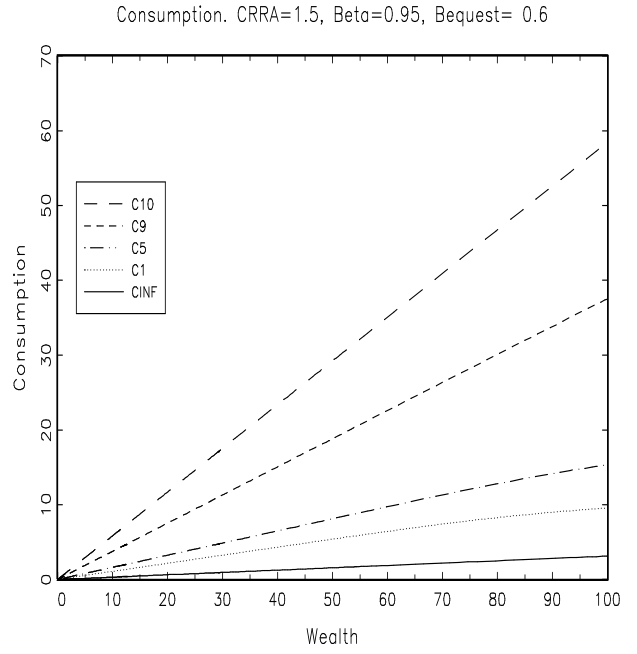


Figure 2: Consumption Decision Rule. CRRA Utility

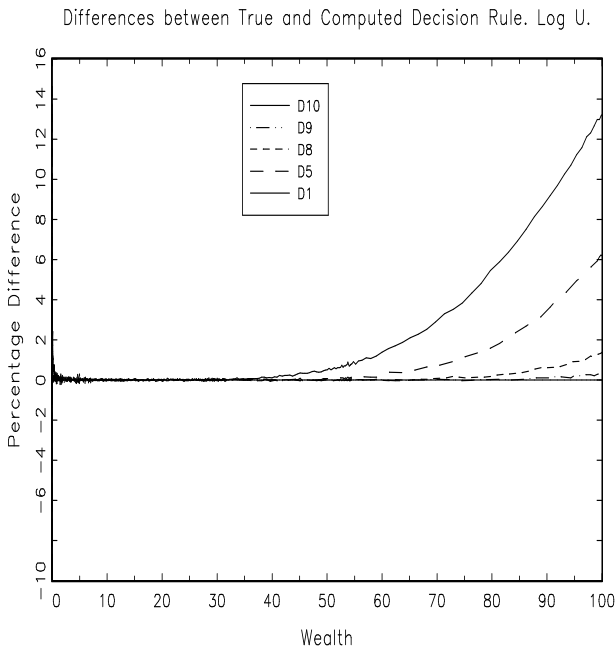


Figure 3: Computed vs. True Decision Rule. Logarithmic Utility

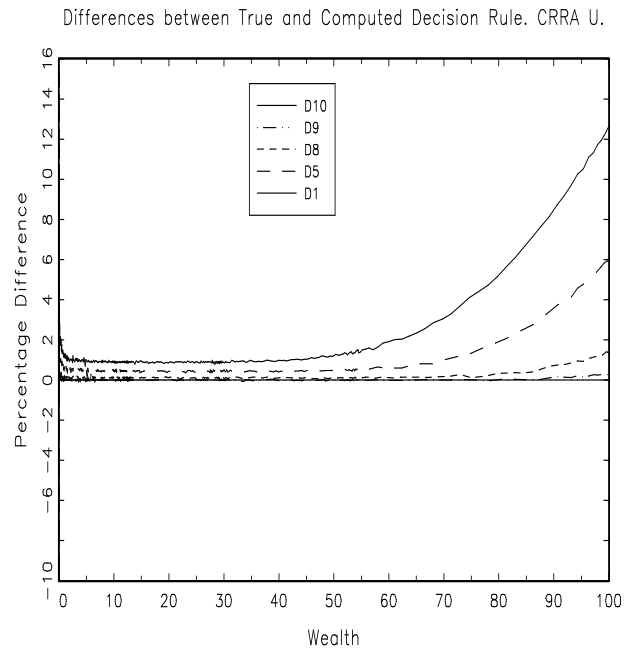


Figure 4: Computed vs. True Decision Rule. CRRA Utility

Figure 5: Simulated Consumption. CRRA Utility

Consumption for C/S Problem, CRRA=1.5, 5000s.

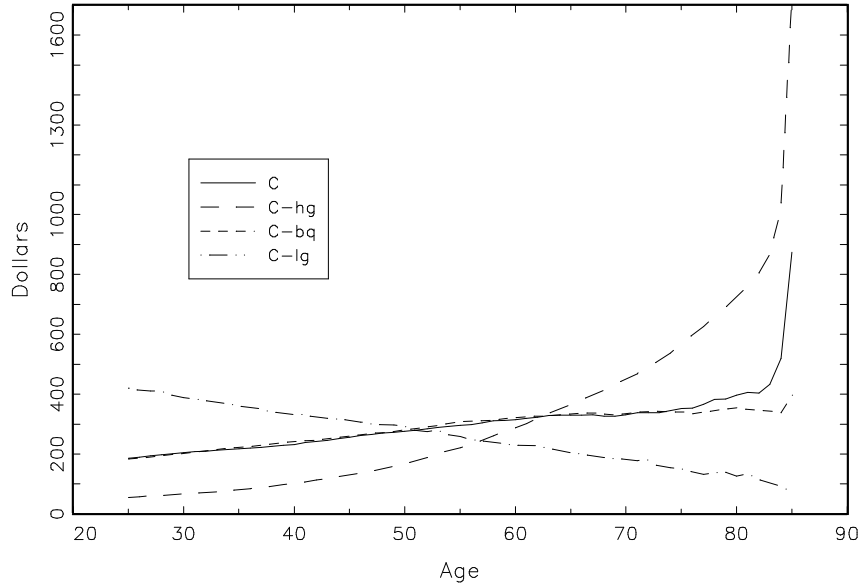


Figure 6: Simulated Wealth Accumulation. CRRA Utility

Wealth Accumulation for C/S Problem, CRRA=1.5, 5000 s.

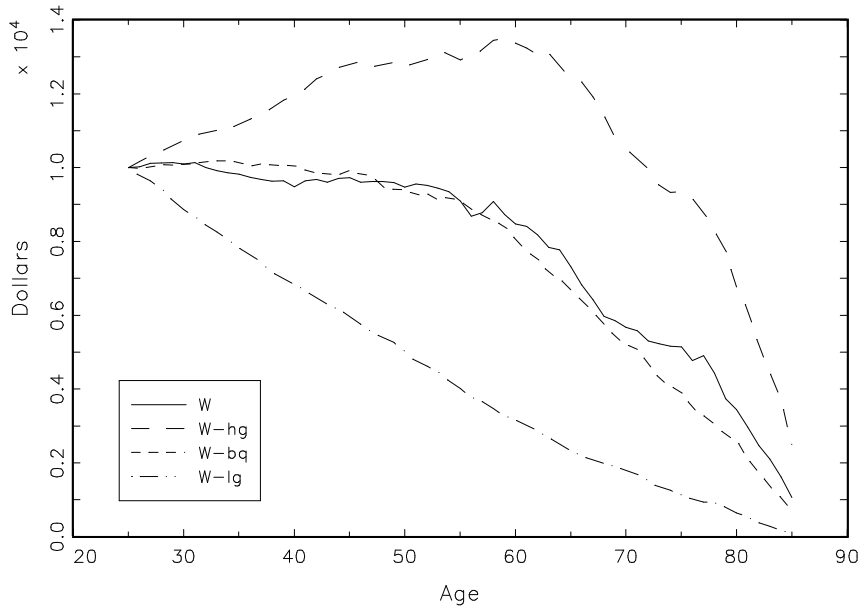


Figure 7: Simulated Consumption. Deterministic Wages

Consumption. Deterministic Wages, CRRA=1.5,2.5, eta=0.7,0.5, 10000s.

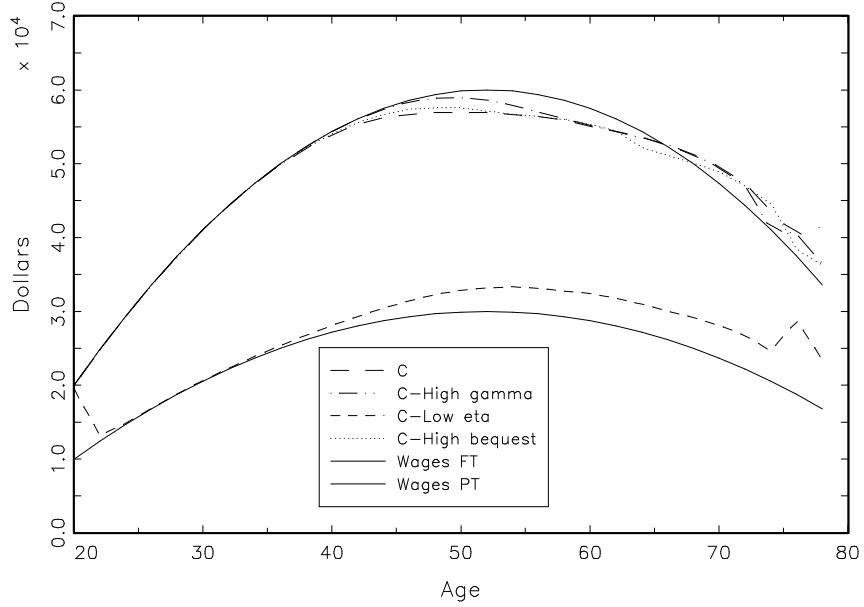


Figure 8: Simulated Labor Supply. Deterministic Wages

Labor. Deterministic Wages, CRRA=1.5,2.5, eta=0.7,0.5, 10000s.

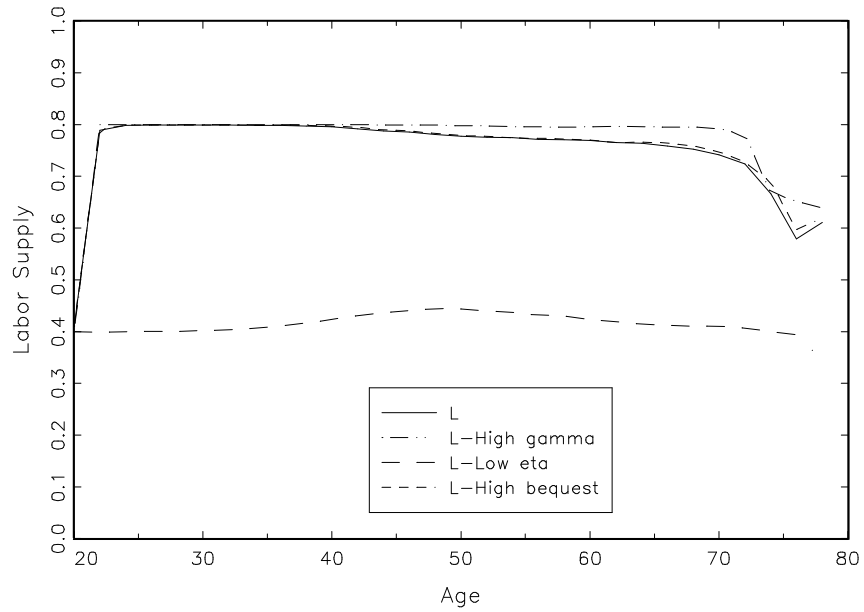


Figure 9: Simulated Wealth. Deterministic Wages

Wealth Path. Deterministic Wages, CRRA=1.5,2.5, eta=0.7,0.5, 10000s.

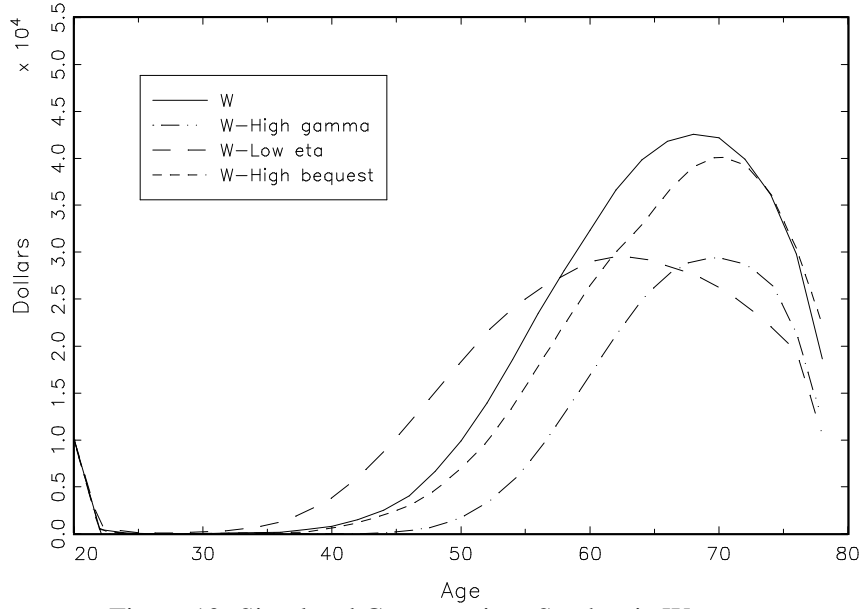


Figure 10: Simulated Consumption. Stochastic Wages

Consumption. Stochastic Wages, CRRA=1.5,2.5, eta=0.7,0.5, 5000s.

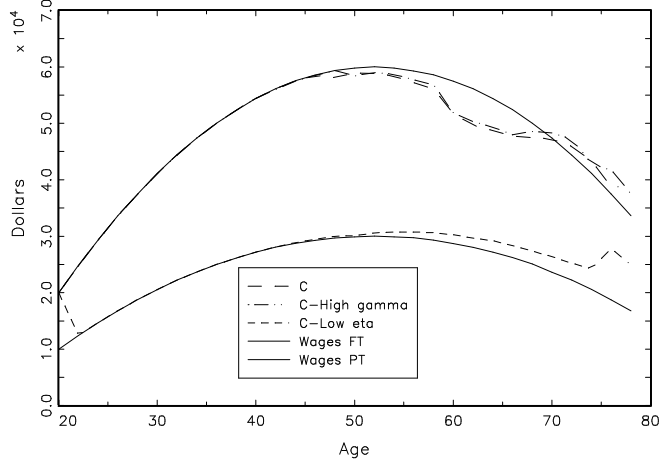


Figure 11: Simulated Labor Supply. Stochastic Wages

Labor. Stochastic Wages, CRRA=1.5,2.5, eta=0.7,0.5, 5000s.

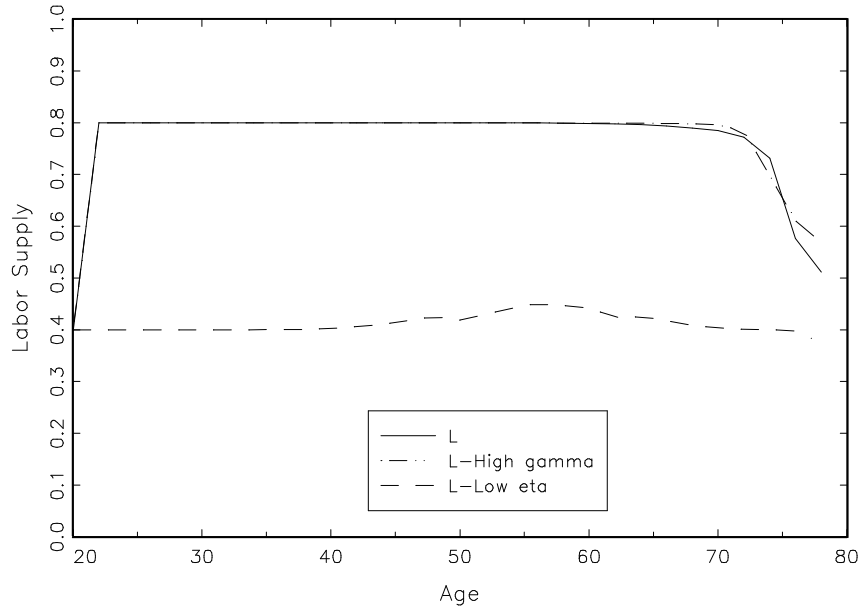


Figure 12: Simulated Wealth. Stochastic Wages

Wealth Path. Stochastic Wages, CRRA=1.5,2.5, eta=0.7,0.5, 5000s.

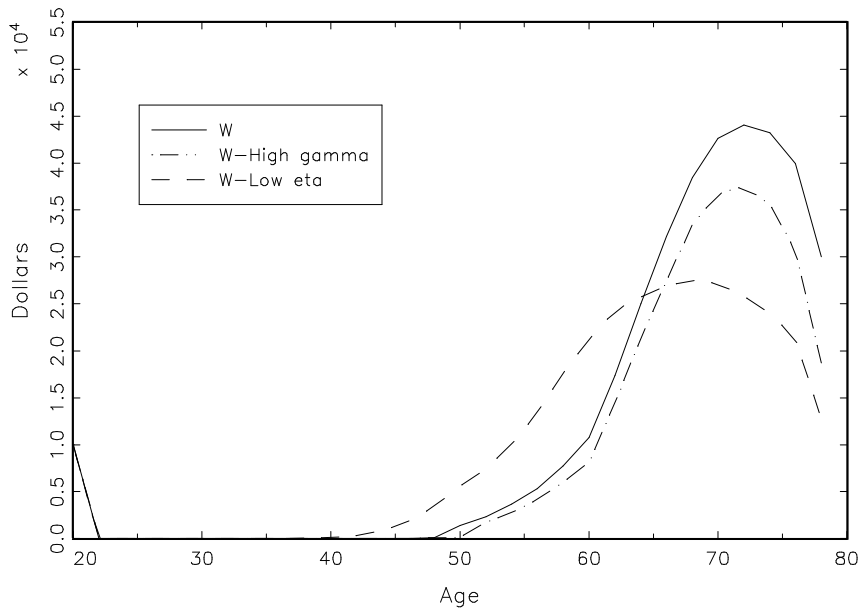


Figure 13: Simulated Consumption. Serially Correlated Wages

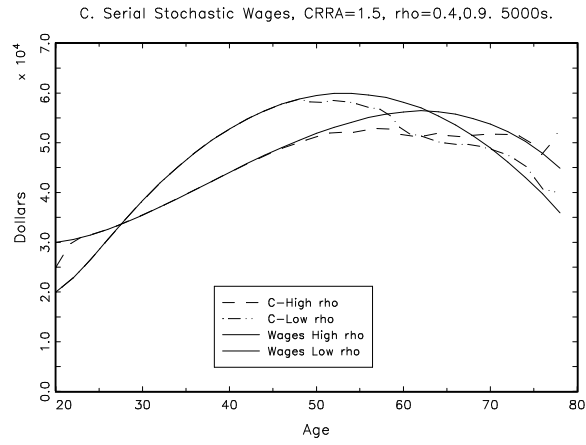


Figure 14: Simulated Labor Supply. Serially Correlated Wages



Figure 15: Simulated Wealth. Serially Correlated Wages

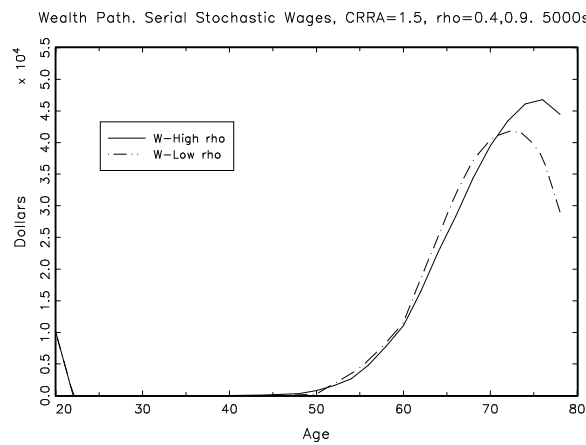


Figure 16: Simulated Consumption. C/S Problem. CRRA Utility

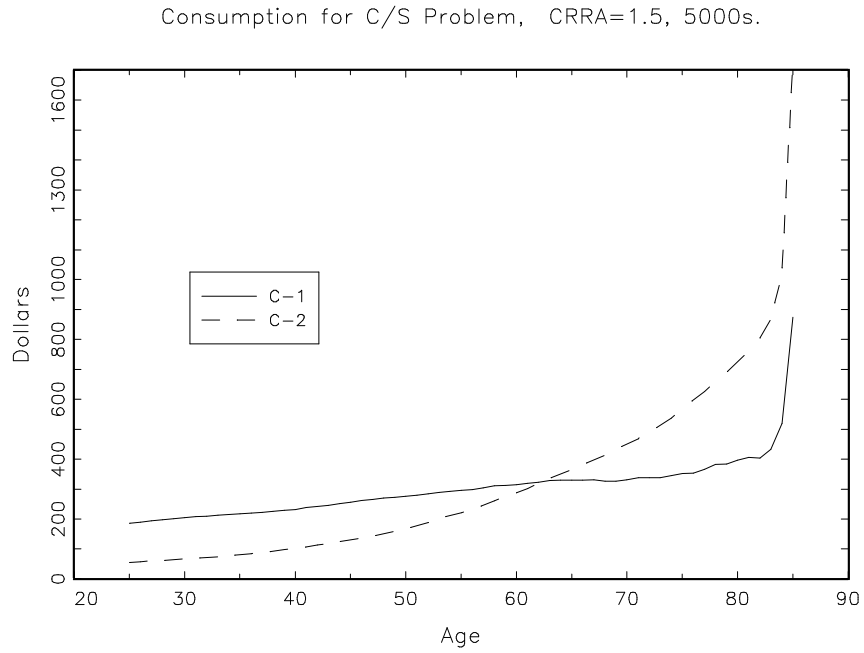


Figure 17: Simulated Consumption and Annuities. C/S Problem. CRRA Utility

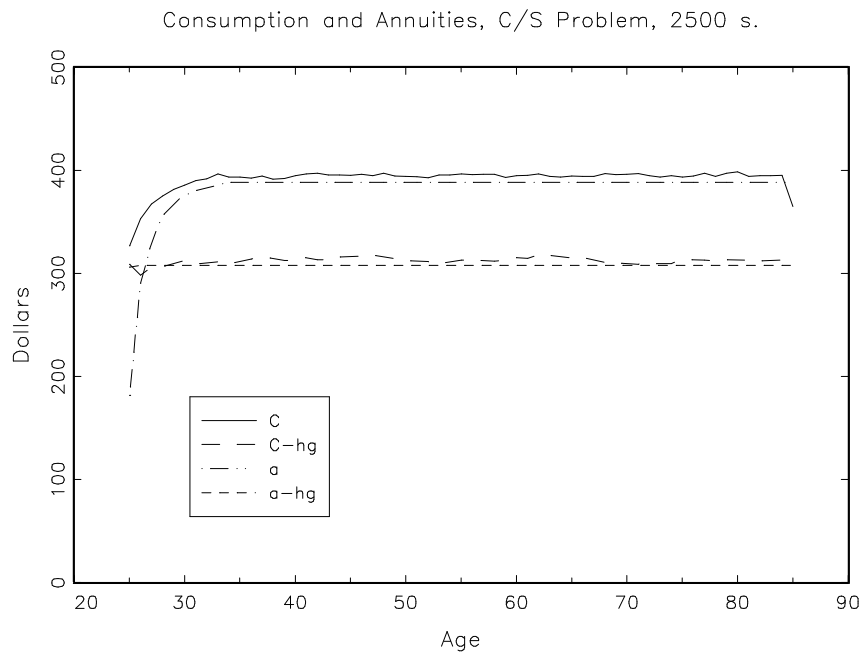


Figure 18: Simulated Wealth Accumulation. C/S Problem. CRRA Utility

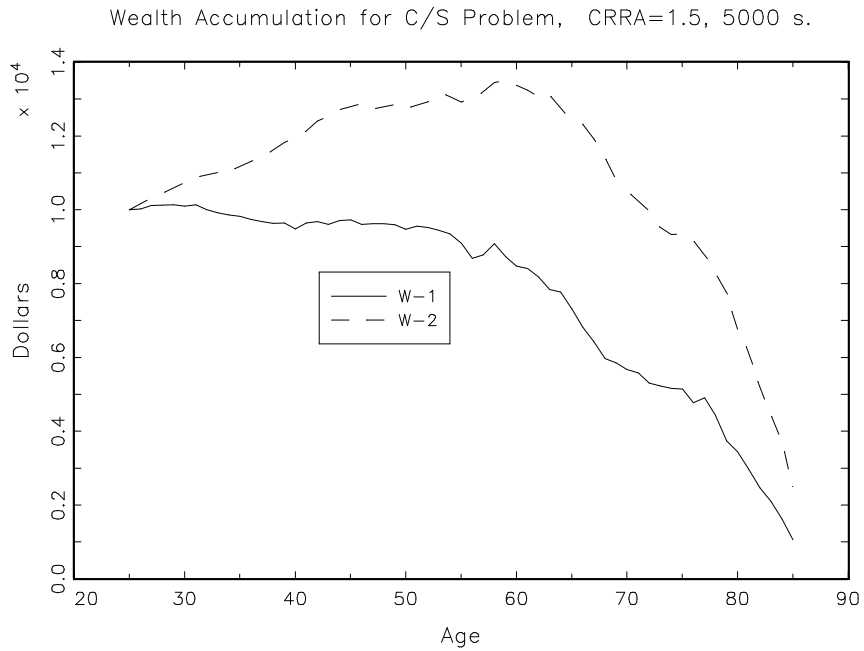


Figure 19: Simulated Wealth and Annuity Costs. C/S Problem. CRRA Utility

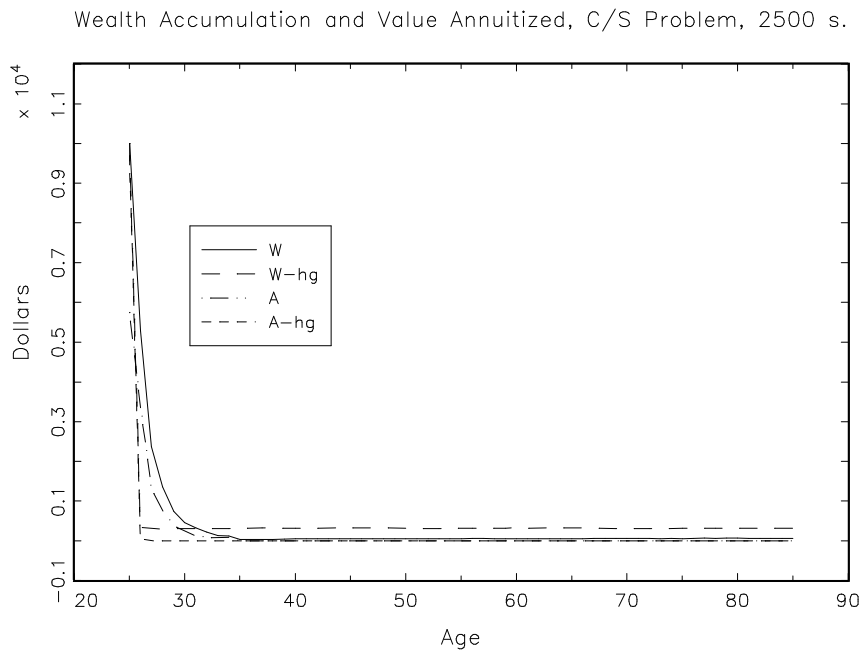


Figure 20: Simulated Consumption and Annuities. Full Model.

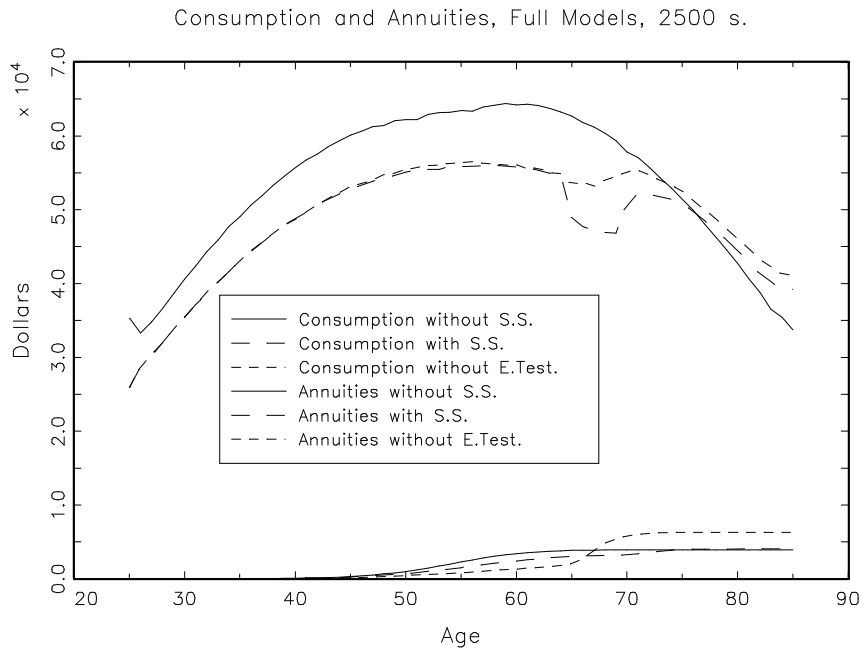


Figure 21: Simulated Labor Supply. Full Model.



Figure 22: Simulated Wealth and Annuity Premiums, without S.S.

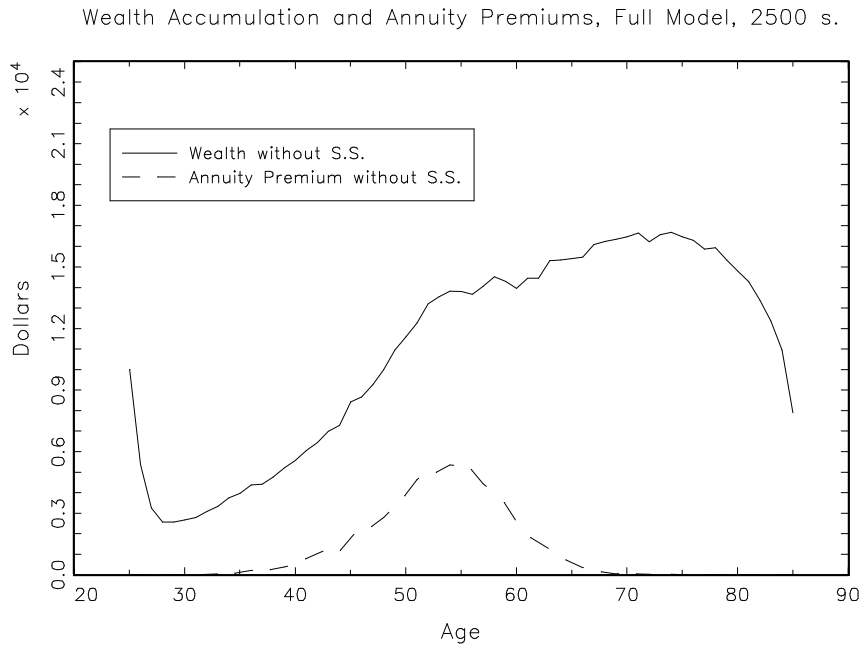


Figure 23: Simulated Wealth and Annuity Premiums, with S.S.

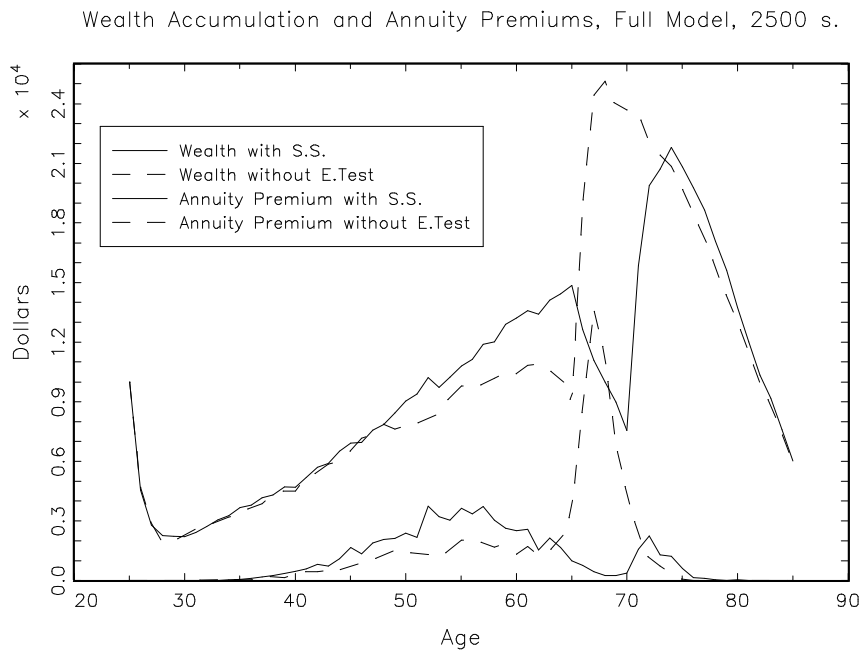


Figure 24: Simulated Wealth and Annuity Premiums, without S.S.

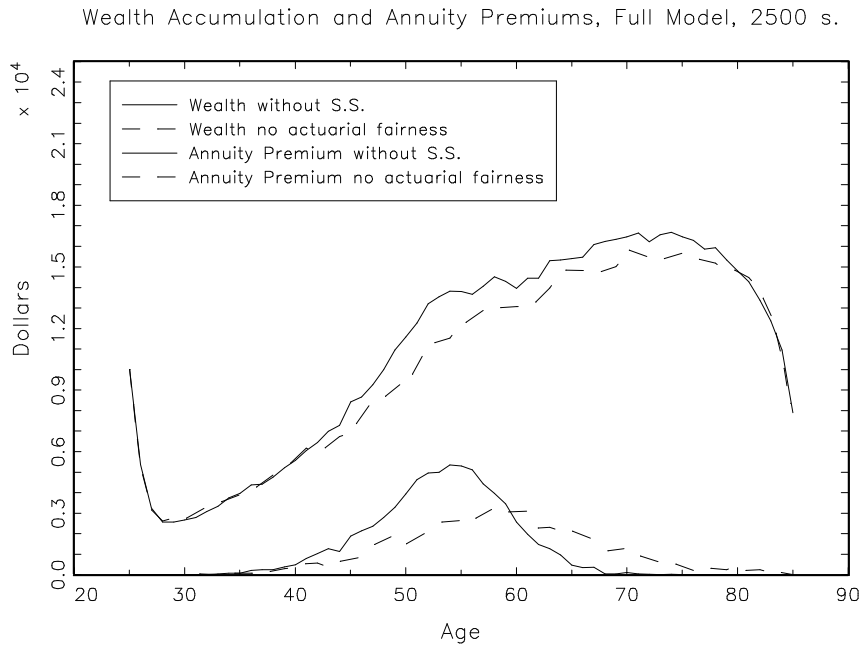
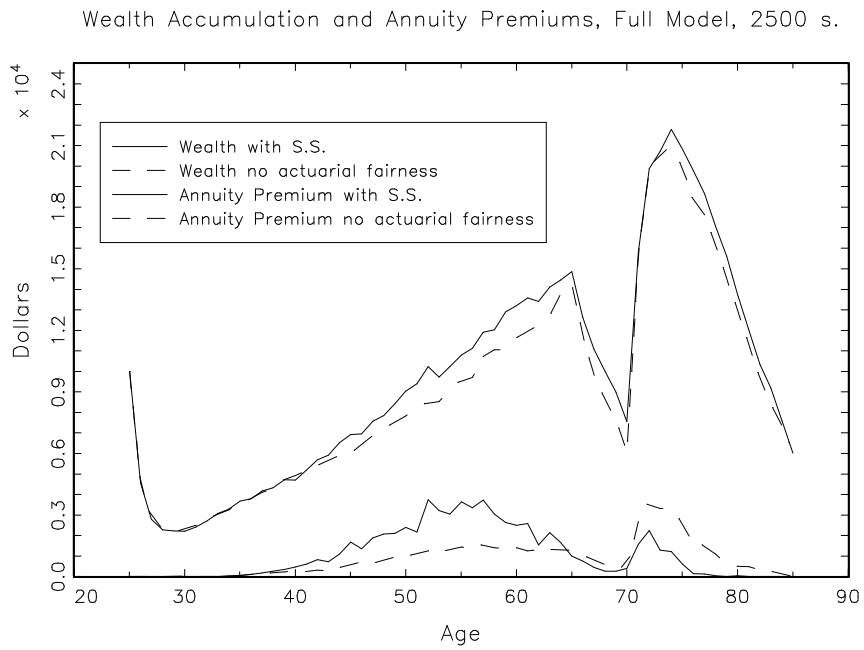


Figure 25: Simulated Wealth and Annuity Premiums, with S.S.



**Table 1. Introducing Annuities. Welfare Comparisons: Equivalent Variation in Percentages.**

Age and Wealth	Consumption/Saving	C/S and Labor/Leisure	C/S, L/L and Social Security
25 and 10,000	63	15	9
25 and 50,000	73	4	2
45 and 10,000	60	49	33
45 and 50,000	68	15	12
65 and 10,000	53	60	50
65 and 50,000	54	31	17

## Appendix

In this Appendix we derive the closed form solution of the finite horizon version of Phelps (1962) consumption/saving problem assuming a CRRA utility function. Our derivation is also close in nature to the one performed in Levhari and Srinivasan (1969). We can again solve this problem relying on Dynamic Programming and Bellman's principle of optimality, using backward induction. In the last period of life agents solve

$$V_T(w) = \max_{0 \leq c \leq w} \frac{c^{1-\gamma}}{1-\gamma} + K \frac{(w-c)^{1-\gamma}}{1-\gamma},$$

where  $\gamma$  is the coefficient of relative risk aversion and  $K$  is the bequest factor, characterized as a number between zero and one.<sup>42</sup> By deriving the first order condition with respect to consumption it is straightforward to show that

$$c_T = \frac{w}{1 + K^{\frac{1}{\gamma}}},$$

we can then write the analytical expression for the last period value function:

$$V_T(w) = \frac{\left(\frac{w}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma} + K \frac{\left(\frac{wK^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma}.$$

Then the problem that agents solve in the next to last period of life is:

$$V_{T-1}(w) = \max_{0 \leq c \leq w} \frac{c^{1-\gamma}}{1-\gamma} + \beta E V_T(w-c).$$

Using the previous results we can write

$$V_{T-1}(w) = \max_{0 \leq c \leq w} \frac{c^{1-\gamma}}{1-\gamma} + \beta E \left[ \frac{\left(\frac{\tilde{r}(w-c)}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma} + K \left[ \frac{\left(\frac{\tilde{r}(w-c)K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma} \right] \right].$$

Here in order to derive the first order condition with respect to consumption we assume, as in Lavhari and Srinivasan (1969), that the value function is differentiable and that the differential and expected value operators can be interchanged. The *f.o.c* is then,

$$c^{-\gamma} - \beta E (\tilde{r}^{1-\gamma}) \left[ \left(\frac{(w-c)}{1+K^{\frac{1}{\gamma}}}\right)^{-\gamma} \frac{1}{1+K^{\frac{1}{\gamma}}} + K \left[ \left(\frac{(w-c)K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right)^{-\gamma} \frac{K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right] \right] = 0.$$

Then some algebraic manipulation allows us to write the *f.o.c* as

$$c^{-\gamma} = \beta E (\tilde{r}^{1-\gamma}) \left(\frac{(w-c)}{1+K^{\frac{1}{\gamma}}}\right)^{-\gamma}.$$

Some more tedious algebra leads to the following expression for the decision rule in the next to last period

$$c_{T-1} = \frac{w}{1 + \beta^{\frac{1}{\gamma}} [E (\tilde{r}^{1-\gamma})]^{\frac{1}{\gamma}} [1 + K^{\frac{1}{\gamma}}]},$$

---

<sup>42</sup> We also follow in this case the "egoistic" model of bequests.

that can be rewritten as

$$c_{T-1} = \frac{w}{1 + \beta^{\frac{1}{\gamma}} [E(\tilde{r}^{1-\gamma})]^{\frac{1}{\gamma}} + \beta^{\frac{1}{\gamma}} [E(\tilde{r}^{1-\gamma})]^{\frac{1}{\gamma}} K^{\frac{1}{\gamma}}}.$$

Assuming next that the interest rate,  $\tilde{r}$ , follows a log-normal distribution with mean  $\mu$  and variance  $\sigma^2$ , then given that  $E(\tilde{r}) = e^{\mu + \frac{\sigma^2}{2}}$  and denoting  $E(\tilde{r})$  as  $\bar{r}$  we can write

$$E(\tilde{r}^{1-\gamma}) = \bar{r}^{1-\gamma} e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}}.$$

We then substitute back in the formula for  $c_{T-1}$  and obtain

$$c_{T-1} = \frac{w}{1 + \beta^{\frac{1}{\gamma}} \left( \bar{r}^{1-\gamma} e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}} \right)^{\frac{1}{\gamma}} + \beta^{\frac{1}{\gamma}} K^{\frac{1}{\gamma}} \left( \bar{r}^{1-\gamma} e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}} \right)^{\frac{1}{\gamma}}},$$

given the similarity with expression (8) in the text it is easy to see how backward induction would lead us to the decision rules for the rest of the periods, for example we can write  $c_{T-k}$  as

$$c_{T-k} = \frac{w}{1 + \beta^{\frac{1}{\gamma}} \left( \bar{r}^{1-\gamma} e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}} \right)^{\frac{1}{\gamma}} + \beta^{\frac{2}{\gamma}} \left( \bar{r}^{1-\gamma} e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}} \right)^{\frac{1}{\gamma}} + \dots + \beta^{\frac{k}{\gamma}} K^{\frac{1}{\gamma}} \left( \bar{r}^{1-\gamma} e^{-\gamma(1-\gamma)\frac{\sigma^2}{2}} \right)^{\frac{1}{\gamma}}}.$$

We can also see that if  $\gamma$  is equal to 1 we are back to the logarithmic utility case and the expression for  $c_{T-1}$  above is equivalent to (8), which is a special case of the expression above. It is also important to emphasize that this expression is the finite horizon counterpart to the one obtained in Levhari and Srinivasan (1969) once a bequest motive is introduced, and that their results regarding the effects of uncertainty (decreasing proportion of wealth consumed as the uncertainty grows if  $\gamma > 1$ ) go through in this case.

## References

- Alessie, R., and A. Lusardi (1997a): "Saving and Income Smoothing: Evidence from Panel Data," *European Economic Review*, **41** 1251–1279.
- Alessie, R., and A. Lusardi (1997b): "Consumption, Saving and Habit Formation," *Economic Letters*, **55** 103–108.
- Alessie, R., A. Lusardi, and T. Aldershof (1997): "Income and Wealth Over the Life Cycle: Evidence from Panel Data," *Review of Income and Wealth*, **43-1** 1–32.
- Ando, A., and F. Modigliani (1963): "The "Life Cycle" Hypothesis of Saving: Aggregate Implications and Tests," *American Economic Review*, **53-1** 55–84.
- Attanasio, O.P., and M. Browning (1995): "Consumption over the Life Cycle and over the Business Cycle," *American Economic Review*, **85-5** 1118–1137.
- Attanasio, O.P., and G. Weber (1995): "Is Consumption Growth Consistent with Intertemporal Optimization? Evidence from the Consumer Expenditure Survey," *Journal of Political Economy*, **103-6** 1121–1157.
- Attanasio, O.P., J. Banks, C. Meghir, and G. Weber (1997): "Humps and Bumps in Lifetime Consumption," manuscript, University College London.
- Banks, J., R. Blundell, and S. Tanner (1998): "Is There a Retirement-Savings Puzzle?" *American Economic Review* **88-4** 769–788.
- Beckmann M.J. (1959): "A Dynamic Programming Model of the Consumption Function," Yale University, Cowles Foundation for Research in Economics, Discussion Paper, **68**.
- Benítez-Silva, H., G. Hall, G.J. Hitsch, G. Pauletto, and J. Rust (2000): "A Comparison of Discrete and Parametric Approximation Methods for Continuous-State Dynamic Programming Problems in Economics," manuscript. SUNY-Stony Brook, Yale University and University of Maryland.
- Benítez-Silva, H. (2002): "Job Search Behavior at the End of the Life Cycle," manuscript. SUNY-Stony Brook.
- Benítez-Silva, H. (2003): "Labor Supply Flexibility and Portfolio Choice: An Empirical Analysis," manuscript. SUNY-Stony Brook.
- Bernheim, B.D. (1987): "Dissaving after Retirement: Testing the Pure Life Cycle Hypothesis," in Z. Bodie, J.B. Shoven, and D.A. Wise (eds.) *Issues in Pension Economics*, 237–274.
- Bernheim, B.D. (1991): "How Strong are Bequest Motives? Evidence Based on Estimates of the Demand for Life Insurance and Annuities," *Journal of Political Economy*, **99-5** 899–927.
- Bodie, Z., and W.F. Samuelson (1989): "Labor Supply Flexibility and Portfolio Choice," *NBER Working Paper Series*, **3043**.
- Bodie, Z., R.C. Merton, and W.F. Samuelson (1992): "Labor supply flexibility and portfolio choice in a life cycle model," *Journal of Economic Dynamics and Control*, **16** 427–449.
- Boskin, M.J., B.D. Bernheim, and P.J. Bayer (1998): *The Economic Role of Annuities*, A Catalyst Institute Research Project. Chicago, Illinois.

- Brown, J.R. (1999a): "Private Pensions, Mortality Risk, and the Decision to Annuitize," *NBER Working Paper Series*, **7191**.
- Brown, J.R. (1999b): "Are the Elderly Really Over-Annuitized? New Evidence on Life Insurance and Bequests," *NBER Working Paper Series*, **7193**.
- Brown, J.R., O.S. Mitchell, J.M. Poterba, and M.J. Warshawsky (2001): *The Role of Annuity Markets in Financing Retirement*. MIT Press: Cambridge, Massachusetts and London, England.
- Browning, M., and A. Lusardi (1996): "Household Saving: Micro Theories and Micro Facts," *Journal of Economic Literature*, **34** 1797–1855.
- Browning, M., and C. Meghir (1991): "The Effects of Male and Female Labor Supply on Commodity Demands," *Econometrica*, **59-4** 925–951.
- Brugiavini, A. (1993): "Uncertainty resolution and the timing of annuity purchases," *Journal of Public Economics*, **50** 31–62.
- Burnside, C. (1999): "Discrete State-Space Methods for the Study of Dynamic Economies," in R. Marimon and A. Scott (eds.), *Computational Methods for the Study of Dynamic Economies*, Oxford University Press, 95–113.
- Bütler, M. (1998): "Anticipation Effects of Looming Public Pension Reforms." Carnegie-Rochester Public Policy Conference.
- Cagetti, M. (1999): "Wealth Accumulation over the Life Cycle and Precautionary Savings," manuscript, University of Chicago.
- Carroll, C.D. (1994): "How Does Future Income Affect Current Consumption?" *Quarterly Journal of Economics*, **109** 111–147.
- Carroll, C.D. (1997): "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis," *Quarterly Journal of Economics*, **112** 1–55.
- Carroll, C.D., and A.A. Samwick (1997): "The Nature of Precautionary Wealth," *Journal of Monetary Economics*, **40** 41–71.
- Cifuentes, R. (1999): "How does Pension Reform affect Savings and Welfare?" manuscript, Harvard University.
- Davies, J.B. (1981): "Uncertain Lifetime, Consumption, and Dissaving in Retirement," *Journal of Political Economy*, **89-3** 561–577.
- Davis, S.J., and P. Willen (2000): "Uncertain Labor Income and Portfolio Choice: Covariance and Its Implications," manuscript, University of Chicago and Princeton University.
- Deaton, A. (1991): "Savings and Liquidity Constraints," *Econometrica*, **59-5** 1221–1248.
- Deaton, A. (1992): *Understanding Consumption*, Clarendon Lectures in Economics, Oxford University Press.
- De Nardi, M., S. İmrohoroğlu, and T.J. Sargent (1999): "Projected U.S. Demographics and Social Security," *Review of Economic Dynamics*, **2** 575–615.
- Dynan, K.E., J. Skinner, and S.P. Zeldes (1999): "Do the Rich Save More?" manuscript, The Federal Reserve Board.

- Eichenbaum, M.S., and D. Peled (1987): "Capital Accumulation and Annuities in an Adverse Selection Economy," *Journal of Political Economy*, **95-2** 334–354.
- Engen, E.M., W.G. Gale, and C.E. Uccello (1999): "The Adequacy of Household Saving," in W.C. Brainard and G.L. Perry (eds.) *Brookings Papers on Economic Activity*, **2**.
- Fama, E.F. (1970): "Multiperiod Consumption-Investment Decision," *American Economic Review*, **60-1** 163–174.
- Feldstein, M. (1974): "Social Security, Induced Retirement, and Aggregate Capital Accumulation," *Journal of Political Economy*, **82-5** 905–926.
- French, E. (2000): "The Labor Supply Response to (Mismeasured but) Predictable Wage Changes," WP 2000-08. Federal Reserve Bank of Chicago.
- French, E. (2000): "The Effects of Health, Wealth, and Wages on Labor Supply and Retirement Behavior," manuscript. University of Wisconsin and Federal Reserve Bank of Chicago.
- Flodén, M. (1998): "Labor Supply and Consumption under Uncertainty: Prudence Reconsidered," manuscript, Stockholm University.
- Friedberg, L. (2000): "The Labor Supply Effects of the Social Security Earnings Test," *Review of Economics and Statistics*, **82-1** 48–63.
- Friedman, M. (1957): *A Theory of the Consumption Function*, Princeton University Press.
- Friedman, B.M., and M.J. Warshawsky (1990): "The Cost of Annuities: Implications for Saving Behavior and Bequests," *Quarterly Journal of Economics*, **105-1** 135–154.
- Geanakoplos, J., O.S. Mitchell, and S. Zeldes (1999): "Social Security Money's Worth," in *Prospects for Social Security Reform*, O.S. Mitchell, R.J. Myers, and H. Young (eds.), University of Pennsylvania Press.
- Gentry, W.M., and J. Milano (1998): "Taxes and Investment in Annuities," *NBER Working Paper Series*, **6525**.
- Gourinchas, P.O., and J.A. Parker (1999): "Consumption over the Life Cycle," *NBER Working Paper Series*, **7271**.
- Hakansson, N.H. (1970): "Optimal Investment and Consumption Strategies Under Risk for a Class of Utility Functions," *Econometrica*, **38** 587–607.
- Heckman, J.J. (1974): "Life Cycle Consumption and Labor Supply: an Explanation of the Relationship between Income and Consumption over the Life Cycle," *American Economic Review*, **64-1** 188–194.
- Heckman J.J. and T.E. MaCurdy (1980): "A Life Cycle Model of Female Labor Supply," *Review of Economic Studies*, **47-1** 47–74.
- Hubbard, R.G. (1987): "Uncertain Lifetimes, Pensions and Individual Saving," in *Issues in Pension Economics*, Z. Bodie, J.B. Shoven and D.A. Wise (eds.). The University of Chicago Press for the NBER.
- Hubbard, R.G., and K.L. Judd (1987): "Social Security and Individual Welfare: Precautionary Saving, Borrowing Constraints, and the Payroll Tax," *American Economic Review*, **77-4** 630–646.

- Hubbard, R.G., J. Skinner, and S.P. Zeldes (1994): "The Importance of Precautionary Motives in Explaining Individual and Aggregate Saving," *Carnegie-Rochester Conference Series on Public Policy*, **40** 59–125.
- Hubbard, R.G., J. Skinner, and S.P. Zeldes (1995): "Precautionary Savings and Social Insurance," *Journal of Political Economy*, **103-2** 360–399.
- Hugget, M., and G. Ventura (1998): "On the Distributional Effects of Social Security Reform," *Review of Economic Dynamics*, **2** 498–531.
- Hurd, M. (1987): "Savings of the Elderly and Desired Bequests," *American Economic Review*, **77-3** 298–312.
- Hurd, M. (1989): "Mortality Risk and Bequests," *Econometrica*, **57-4** 779–813.
- İmrohoroğlu, A., S. İmrohoroğlu, and D.H. Joines (1994): "The Effect of Tax-Favored Retirement Accounts on Capital Accumulation and Welfare," discussion paper, Institute for Empirical Macroeconomics, Federal Reserve Bank of Minneapolis.
- İmrohoroğlu, A., S. İmrohoroğlu, and D.H. Joines (1999a): "Computing Models of Social Security," in R. Marimon and A. Scott (eds.), *Computational Methods for the Study of Dynamic Economies*, Oxford University Press, 221–237.
- İmrohoroğlu, A., S. İmrohoroğlu, and D.H. Joines (1999b): "Myopia and Social Security," manuscript, University of Southern California, Marshall School of Business.
- Judd, K.L. (1998): *Numerical Methods in Economics*, The MIT Press.
- Kotlikoff, L.J. (1979): "Testing the Theory of Social Security and Life Cycle Accumulation," *American Economic Review*, **69-3** 396–410.
- Kotlikoff, L.J., and A. Spivak (1981): "The Family as an Incomplete Annuities Market," *Journal of Political Economy*, **89-2** 372–391.
- Kotlikoff, L.J., and L.H. Summers (1981): "The Role of Intergenerational Transfers in Aggregate Capital Accumulation," *Journal of Political Economy*, **89-4** 706–732.
- Kotlikoff, L.J., J.B. Shoven, and A. Spivak (1987): "Annuity Markets, Savings, and the Capital Stock," in *Issues in Pension Economics*, Z. Bodie, J.B. Shoven and D.A. Wise (eds.). The University of Chicago Press for the NBER.
- Laitner, J., and T. Juster (1996): "New Evidence on Altruism: A Study of TIAA-CREF Retirees," *American Economic Review*, **86-4** 893–908.
- Levhari, D., and T.N. Srinivasan (1969): "Optimal Savings under Uncertainty," *Review of Economic Studies*, **36-2** 153–163.
- Low, H.W. (1998): "Simulation Methods and Economic Analysis," manuscript, University College London, University of London.
- Low, H.W. (1999): "Self Insurance and Unemployment Benefit in a Life-Cycle Model of Labour Supply and Savings," manuscript, University of Cambridge.
- Lusardi, A. (1998): "On the Importance of the Precautionary Saving Motive," *American Economic Review*, **88-2** 449–453.

- MaCurdy, T.E. (1981): “An Empirical Model of Labor Supply in a Life-Cycle Setting,” *Journal of Political Economy*, **89-6** 1059–1085.
- MaCurdy, T.E. (1983): “A Simple Scheme for Estimating and Intertemporal Model of Labor Supply and Consumption in the Presence of Taxes and Uncertainty,” *International Economic Review*, **24-2** 265–289.
- Merton, R.C. (1969): “Lifetime Portfolio Selection Under Uncertainty: the Continuous-Time Case,” *Review of Economics and Statistics*, **51-3** 247–257.
- Miller, B. L. (1974): “Optimal Consumption with a Stochastic Income Stream,” *Econometrica*, **42-2** 253–266.
- Mitchell, O.S., J.M. Poterba, M.J. Warshawsky, and J.R. Brown (1999): “New Evidence on the Money’s Worth of Individual Annuities,” *American Economic Review*, **89-5** 1299-1318.
- Modigliani, F., and R. Brumberg (1954): “Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data,” in K. Kurihara (ed.) *Post-Keynesian Economics*, Rutgers University Press.
- Modigliani, F., and R. Brumberg (1980): “Utility Analysis and Aggregate Consumption Functions: An Attempt at Integration,” in *The Collected Papers of Franco Modigliani* Vol. 2. MIT Press.
- Modigliani, F. (1988): “The Role of Intergenerational Transfers and Life Cycle Saving in the Accumulation of Wealth,” *Journal of Economic Perspectives*, **2-2** 15–40.
- Myers, R.J. (1993): *Social Security*, Pension Research Council, University of Pennsylvania Press.
- Nagatani, K. (1972): “Life Cycle Saving: Theory and Fact,” *American Economic Review*, **62-3** 344–353.
- Page, B. (1998): “Social Security and Private Saving: A Review of the Empirical Evidence,” manuscript, Congressional Budget Office.
- Palumbo, M.G. (1999): “Uncertain Medical Expenses and Precautionary Saving Near the End of the Life Cycle,” *Review of Economic Studies*, **66-2** 395–421.
- Phelps, E.S. (1962): “The Accumulation of Risky Capital: A Sequential Utility Analysis,” *Econometrica*, **30** 729–743.
- Poterba, J.M. (1998): “Population Age Structure and Asset Returns: An Empirical Investigation,” *NBER Working Paper Series*, **6774**.
- Press, W.H., S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery (1992): *Numerical Recipes in Fortran*, Cambridge University Press.
- Rust, J. (1996): “Numerical Dynamic Programming in Economics,” in *Handbook of Computational Economics*. H. Amman et al. (eds.) Amsterdam: Elsevier.
- Rust, J. (1997): “A Comparison of Policy Iteration Methods for Solving Continuous-State, Infinite-Horizon Markovian Decision Problems Using Random, Quasi-random, and Deterministic Discretizations,” manuscript, University of Maryland.
- Rust, J. (1999a): “Solving High Dimensional Problems via Decentralization,” Invited Lecture in Computational Economics. Latin American Meetings of the Econometric Society, Cancun, Mexico.

- Rust, J. (1999b): "Strategies for Incorporating Risk, Uncertainty, and Private Insurance Mechanisms in Models of Social Insurance," manuscript, University of Maryland.
- Rust, J. and C. Phelan (1997): "How Social Security and Medicare Affect Retirement Behavior in a World of Incomplete Markets," *Econometrica*, **65-4** 781–831.
- Rust, J., M. Buchinsky, and H. Benítez-Silva (2002): "Dynamic Structural Models of Retirement and Disability," manuscript. University of Maryland, UCLA, and SUNY-Stony Brook.
- Samuelson, P.A. (1969): "Lifetime Portfolio Selection By Dynamic Stochastic Programming," *Review of Economics and Statistics*, **51-3** 239–246.
- Sandmo, A. (1970): "The Effect of Uncertainty on Saving Decisions," *Review of Economic Studies*, **37-3** 353–360.
- Seater, J.J. (1977): "A Unified Model of Consumption, Labor Supply, and Job Search," *Journal of Economic Theory*, **14** 349–372.
- Sheshinski, E. (1999): "Annuities and Retirement," manuscript, The Hebrew University of Jerusalem.
- Siven, C-H. (1974): "Consumption, Supply of Labor and Search Activity in an Intertemporal Perspective," *Swedish Journal of Economics*, **76-1** 44–61.
- Skinner, J. (1988): "Risky Income, Life Cycle Consumption, and Precautionary Savings," *Journal of Monetary Economics*, **22** 237–255.
- Social Security Administration (2000): "How Your Retirement Benefit Is Figured," at [www.ssa.gov](http://www.ssa.gov)
- Tauchen, G., and R. Hussey (1991): "Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models," *Econometrica*, **59-2** 371–396.
- Thurow, L. (1969): "The Optimum Lifetime Distribution of Consumption Expenditures," *American Economic Review*, **59-3** 324–330.
- TIAA-CREF (1999): *Research Dialogues*, **60** TIAA-CREF Institute, New York.
- Walliser, J. (1997): "Understanding Adverse Selection in the Annuities Market and the Impact of Privatizing Social Security," manuscript, Congressional Budget Office.
- Walliser, J. (1998): "Social Security Privatization and the Annuities Market," manuscript, Congressional Budget Office.
- Wilhelm, M.O. (1996): "Bequest Behavior and the Effect of Heirs' Earnings: Testing the Altruistic Model of Bequests," *American Economic Review*, **86-4** 874–892.
- Yaari, M.E. (1965): "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer," *Review of Economic Studies*, **32-2** 137–150.
- Zeldes, S.P. (1989a): "Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence," *Quarterly Journal of Economics*, **104-2** 275–298.
- Zeldes, S.P. (1989b): "Consumption and Liquidity Constraints: An Empirical Investigation," *Journal of Political Economy*, **97-2** 305–346.